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Generalized reflection root systems

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ABSTRACT

We study a combinatorial object, which we call a GRRS (generalized reflection root system); the classical root systems and GRSs introduced by V. Serganova are examples of finite GRRSs. A GRRS is finite if it contains a finite number of vectors and is called affine if it is infinite and has a finite minimal quotient. We prove that an irreducible GRRS containing an isotropic root is either finite or affine; we describe all finite and affine GRRSs and classify them in most of the cases.

0. Introduction

We study a combinatorial object, which we call a GRRS (generalized reflection root system), see Definition 1.2. The classical root systems are finite GRRSs without isotropic roots. Our definition of GRRS is motivated by Serganova's definition of GRS (generalized root system) introduced in [6], Sect. 1, and by the following examples: the set of real roots Δ_{re} of a symmetrizable Kac–Moody superalgebra introduced in [3], [7] and its subsets $\Delta_{re}(\lambda)$ ("integral real roots"), see [1].

Each GRRS R is, by definition, a subset of a finite-dimensional complex vector space V endowed with a symmetric bilinear form $(-,-)$. The image of R in $V/Ker(-,-)$ is

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denoted by $cl(R)$; it satisfies weaker properties than GRRS and is called a WGRS (weak generalized root system). An infinite GRRS is called *affine* if its image $cl(R)$ is finite (in this case $cl(R)$ is a finite WGRS, which were classified in [6]). We show that an irreducible GRRS containing an isotropic root is either finite or affine. Recall a theorem of C. Hoyt that a symmetrizable Kac–Moody superalgebra with an isotropic simple root and an indecomposable Cartan matrix (this corresponds to the irreducibility of GRRS) is finite-dimensional or affine, see [2].

Finite GRRSs correspond to the root systems of finite-dimensional Kac–Moody superalgebras. In this paper we describe all affine GRRSs R and classify them for most cases of $cl(R)$. Irreducible affine GRRSs with $\dim Ker(-,-) = 1$ correspond to symmetrizable affine Lie superalgebras. This case was treated in [9]; in particular, it implies that an "irreducible subsystem" of the set of real roots of an affine Kac–Moody superalgebra is a set of real roots of an affine or a finite-dimensional Kac–Moody superalgebra (this was used in [1]).

For each GRRS we introduce a certain subgroup of $AutR$, which is denoted by $GW(R)$; if R does not contain isotropic roots, then $GW(R)$ is the usual Weyl group. Let R be an irreducible affine GRRS (i.e., $cl(R)$ is finite). We show that if the action of $GW(cl(R))$ on $cl(R)$ is transitive and $cl(R) \neq A_1$, then R is either the affinization of $cl(R)$ (see § 1.7 for definition) or, if $cl(R)$ is the root system of $\mathfrak{psl}(n,n)$, $n > 2$, R is a certain "bijective quotient" of the affinization of the root system of $\mathfrak{pgl}(n,n)$, see § 1.5 for definition. The action of $GW(cl(R))$ on $cl(R)$ is transitive if and only if $cl(R)$ is the root system of a simply laced Lie algebra or a Lie superalgebra $\mathfrak{g} \neq B(m,n)$, which is not a Lie algebra. If R is such that $cl(R) = B(m,n)$, $m,n \geq 1$ or $cl(R) = B_n, C_n$, $n \geq 3$, then R is classified by non-empty subsets of the affine space $\mathbb{F}_2^k$ up to affine automorphisms of $\mathbb{F}_2^k$, where $\dim Ker(-,-) = k$. A similar classification holds for $cl(R) = A_1$. In the cases $cl(R) = G_2, F_4$, the GRRSs R are parametrized by $s = 0, 1, \ldots, \dim Ker(-,-)$. In the remaining case either $cl(R)$ is a finite WGRS, which is not a GRRS, or $cl(R) = BC_n$. We partially classify the corresponding GRRSs (we describe all possible R).

Another combinatorial object, an extended affine root supersystem (EARS), was introduced and described in a recent paper of M. Yousofzadeh [12]. The main differences between a GRRS and an EARS are the following: first, EARS has a "string property", namely, for each α, β in an EARS with $(\alpha,\alpha) \neq 0$ the intersection of $\beta + \mathbb{Z}\alpha$ with the EARS is a string $\{\beta - j\alpha |\ j \in \{-p, p+1, \ldots, q\}\}$ for some $p, q \in \mathbb{Z}$ with $p - q = 2(\alpha,\beta)/(\alpha,\alpha)$. Second, a GRRS should be invariant with respect to the "reflections" connected to its elements. The string property implies the invariance with respect to the reflections connected to non-isotropic roots (α such that $(\alpha,\alpha) \neq 0$). A finite GRRS corresponds to the root system of a finite-dimensional Kac–Moody superalgebra, and the finite EARSs include two additional series. The root system of a symmetrizable affine Lie superalgebra is an EARS and the set of real roots is a GRRS. Moreover, the set of roots of a symmetrizable Kac–Moody superalgebra is an EARS only if this algebra is affine or finite-dimensional (by contrast, the set of real roots is always a GRRS). For example, the real roots of a Kac–Moody algebra with the Cartan