Journal of Algebra 491 (2017) 573–610

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Drinfeld orbifold algebras for symmetric groups

B. Foster-Greenwood [a,b,*], C. Kriloff [a]

[a] *Department of Mathematics and Statistics, Idaho State University, Pocatello, ID 83209, USA*
[b] *Department of Mathematics and Statistics, California State Polytechnic University, Pomona, CA 91768, USA* [1]

A R T I C L E I N F O

Article history:
Received 29 November 2016
Available online 30 August 2017
Communicated by Jon Carlson

MSC:
primary 16S80
secondary 16E40, 16S35, 20B30

Keywords:
Skew group algebra
Deformations
Drinfeld orbifold algebra
Hochschild cohomology
Poincaré–Birkhoff–Witt conditions
Symmetric group

A B S T R A C T

Drinfeld orbifold algebras are a type of deformation of skew group algebras generalizing graded Hecke algebras of interest in representation theory, algebraic combinatorics, and noncommutative geometry. In this article, we classify all Drinfeld orbifold algebras for symmetric groups acting by the natural permutation representation. This provides, for nonabelian groups, infinite families of examples of Drinfeld orbifold algebras that are not graded Hecke algebras. We include explicit descriptions of the maps recording commutator relations and show there is a one-parameter family of such maps supported only on the identity and a three-parameter family of maps supported only on 3-cycles and 5-cycles. Each commutator map must satisfy properties arising from a Poincaré–Birkhoff–Witt condition on the algebra, and our analysis of the properties illustrates reduction techniques using orbits of group element factorizations and intersections of fixed point spaces.

* Corresponding author at: Department of Mathematics and Statistics, California State Polytechnic University, Pomona, CA 91768, USA.
E-mail addresses: brianaf@cpp.edu (B. Foster-Greenwood), krilcath@isu.edu (C. Kriloff).

[1] Permanent address.

http://dx.doi.org/10.1016/j.jalgebra.2017.07.027

1. Introduction

Numerous algebras of intense recent study and interest arise as deformations of skew group algebras $S(V)\#G$, where G is a finite group acting linearly on a finite-dimensional vector space V and $S(V)$ is the symmetric algebra. A grading on the skew group algebra is determined by assigning degree one to vectors in V and degree zero to elements of the group algebra. Drinfeld graded Hecke algebras are constructed by identifying commutators of elements of V with carefully chosen elements of degree zero (i.e., from the group algebra) to yield a deformation of the skew group algebra. In [19], Drinfeld orbifold algebras are similarly defined but additionally allow for degree-one terms in the commutator relations. The resulting algebras are also deformations of the skew group algebra.

Besides capturing a new realm of deformations of skew group algebras, Drinfeld orbifold algebras encompass many known algebras of interest in representation theory, noncommutative geometry, and mathematical physics. The term "Drinfeld orbifold algebras" alludes to the subject's origins in [3], where Drinfeld introduced a broad class of algebras to serve as noncommutative coordinate rings for singular orbifolds. When the group is a Coxeter group acting by its reflection representation, Drinfeld's algebras are isomorphic (see [16]) to the graded Hecke algebras from [12], which arise from a filtration of an affine Hecke algebra when the group is crystallographic (see [13]). The representation theory of these algebras is useful in understanding representations and geometric structure of reductive p-adic groups.

A recent focus on symplectic reflection algebras, which are Drinfeld Hecke algebras for symplectic reflection groups acting on a symplectic vector space, began with [5]. The importance of these algebras lies in the fact that the center of the skew group algebra is the ring of invariants, $\mathbb{C}[V]^G = \mathrm{Spec}(V/G)$, and in the philosophy that the center of a deformation of the skew group algebra may then deform $\mathbb{C}[V]^G$ (see the surveys [9,1]). As a special case, rational Cherednik algebras arise by pairing a reflection representation with its dual and are related to integrable Calogero–Moser systems in physics and deep results in combinatorics (see for instance the surveys [10,4]).

Drinfeld orbifold algebras afford two advantageous views: as quotient algebras satisfying a Poincaré–Birkhoff–Witt (PBW) condition and as formal algebraic deformations of skew group algebras. While PBW conditions relate an algebra to homogeneous shadows of itself that have well-behaved bases, algebraic deformation theory (à la Gerstenhaber [7]) focuses on how the multiplicative structure varies with a deformation parameter and provides a framework of understanding via Hochschild cohomology. In particular, every formal deformation arises from a Hochschild 2-cocycle.

Fruitful techniques arise from a melding of the PBW perspective with the deformation theory perspective (see the survey [21]). Braverman and Gaitsgory [2] and also Polishchuk and Positelski [15] initiated the use of homological methods to study PBW conditions in the context of quadratic algebras of Koszul type. Etingof and Ginzburg applied some of these ideas in an expanded setting in their seminal paper on symplectic reflection