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On the discriminant of twisted tensor products

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ABSTRACT

We provide formulas for computing the discriminant of noncommutative algebras over central subalgebras in the case of Ore extensions and skew group extensions. The formulas follow from a more general result regarding the discriminants of certain twisted tensor products. We employ our formulas to compute automorphism groups for examples in each case.

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1. Introduction

Throughout $\Bbbk$ is an algebraically closed, characteristic zero field and all algebras are $\Bbbk$-algebras. All unadorned tensor products should be regarded as over $\Bbbk$. Given an algebra R, we denote by $R^\times$ the set of units in R. If $\sigma \in \operatorname{Aut}(R)$, then R^σ denotes the subalgebra of elements of R that are fixed under σ. We denote the center of R by $C(R)$.

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Automorphism groups of commutative and noncommutative algebras can be notoriously difficult to compute. For example, $\mathrm{Aut}(\Bbbk[x,y,z])$ is not yet fully understood. In [2], the authors give a method for determining the automorphism groups of noncommutative algebras using the discriminant. This was studied further in [3–6]. Discriminants of deformations of polynomial rings were computed using Poisson geometry in [11,13].

We refer the reader to [2] for the general definitions of trace and discriminant in the context of noncommutative algebras. We review the definitions only in the case that B is an algebra finitely generated free over a central subalgebra $R \subseteq C(B)$ of rank n.

Left multiplication defines a natural embedding $\mathrm{lm} : B \to \mathrm{End}_C(B) \cong M_n(R)$. The usual matrix trace defines a map $\mathrm{tr}_{\mathrm{int}} : M_n(R) \to R$ called the internal trace. The regular trace is defined as the composition $\mathrm{tr}_{\mathrm{reg}} : B \xrightarrow{\mathrm{lm}} M_n(R) \xrightarrow{\mathrm{tr}_{\mathrm{int}}} R$. For our purposes, tr will be $\mathrm{tr}_{\mathrm{reg}}$.

Let ω be a fixed integer and $Z := \{z_i\}_{i=1}^{\omega}$ a subset of B. The discriminant of Z is defined to be

$$d_\omega(Z) = \det(\mathrm{tr}(z_i z_j))_{\omega\times\omega} \in R.$$

If Z is an R-basis of B, then the discriminant of B over R is defined to be

$$d(B/R) =_{R^\times} d_\omega(Z),$$

where $x =_{R^\times} y$ means $x = cy$ for some $c \in R^\times$.

The discriminant is independent of R-linear bases of B [2, Proposition 1.4]. Moreover, if $\phi \in \mathrm{Aut}(B)$ and ϕ preserves R, then ϕ preserves the ideal generated by $d(B/R)$ [2, Lemma 1.8].

Computing the discriminant is a computationally difficult task, even for algebras with few generators. For example, the matrix obtained from $\mathrm{tr}(z_i z_j)$ for the skew group algebra $\Bbbk_{-1}[x_1, x_2, x_3]\#\mathcal{S}_3$ has size 288×288. Our first goal is to provide methods for obtaining the discriminant in cases where the algebra may be realized as an extension of a smaller algebra where computations may be easier.

If A is an algebra and $\sigma \in \mathrm{Aut}(A)$, then the Ore extension $A[t;\sigma]$ is generated by A and t with the rule $ta = \sigma(a)t$ for all $a \in A$.

Theorem 1 *(Theorem 6.1). Let A be an algebra and set $S = A[t;\sigma]$, where $\sigma \in \mathrm{Aut}(A)$ has order $m < \infty$ and no σ^i, $1 \le i < m$, is inner. Suppose R is a central subalgebra of S and set $B = R \cap A^\sigma$. If A is finitely generated free over B of rank n and $R = B[t^m]$, then S is finitely generated free over R and*

$$d(S/R) =_{R^\times} (d(A/B))^m \left(t^{m-1}\right)^{mn}.$$

We say an automorphism σ of A is inner if there exists $a \in A$ such that $xa = a\sigma(x)$ for all $x \in A$. This is not the standard definition of an inner automorphism but it agrees if a

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