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Globally irreducible Weyl modules

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In the representation theory of split reductive algebraic groups, it is well known that every Weyl module with minuscule highest weight is irreducible over every field. Also, the adjoint representation of E_8 is irreducible over every field. In this paper, we prove a converse to these statements, as conjectured by Gross: if a Weyl module is irreducible over every field, it must be either one of these, or trivially constructed from one of these. We also prove a related result on non-degeneracy of the reduced Killing form.

1. Introduction

Split semisimple linear algebraic groups over arbitrary fields can be viewed as a generalization of semisimple Lie algebras over the complex numbers, or even compact real

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Lie groups. As with Lie algebras, such algebraic groups are classified up to isogeny by their root system. Moreover, the set of irreducible representations of such a group is in bijection with the cone of dominant weights for the root system and the representation ring — i.e., K_0 of the category of finite-dimensional representations — is a polynomial ring with generators corresponding to a basis of the cone.

One way in which this analogy breaks down is that, for an algebraic group G over a field k of *prime* characteristic, in addition to the irreducible representation $L(\lambda)$ corresponding to a dominant weight λ, there are three other representations naturally associated with λ, namely the standard module $H^0(\lambda)$, the Weyl module $V(\lambda)$, and the tilting module $T(\lambda)$.[3] The definition of $H^0(\lambda)$ is particularly simple: view k as a one-dimensional representation of a Borel subgroup B of G where B acts via the character λ, then define $H^0(\lambda) := \mathrm{ind}_B^G \lambda$ to be the induced G-module. The *Weyl module* $V(\lambda)$ is the dual of $H^0(-w_0\lambda)$ for w_0 the longest element of the Weyl group and has head $L(\lambda)$. Typical examples of Weyl modules are $\mathrm{Lie}(G)$ for G semisimple simply connected ($V(\lambda)$ for λ the highest root) and the natural module of SO_n. See [22] for general background on these three families of representations.

It turns out that if any two of the four representations $L(\lambda)$, $H^0(\lambda)$, $V(\lambda)$, $T(\lambda)$ are isomorphic over a given field k, then all four are. Our focus is on the question: for which λ are all four isomorphic for *every* field k?

This can be interpreted as a question about representations of split reductive group schemes over $\mathbb{Z}$. Recall that isomorphism classes of such groups are in bijection with (reduced) root data as described in [8, XXIII.5.2]. A root datum for a group G includes a character lattice $X(T)$ of a split maximal torus T and the set $R \subset X(T)$ of roots of G with respect to T. Picking an ordering on R specifies a cone of dominant weights $X(T)_+$ in $X(T)$. For each $\lambda \in X(T)_+$, there is a representation $V(\lambda)$ for G, defined over $\mathbb{Z}$, that is generated by a highest weight vector with weight λ such that $V(\lambda) \otimes \mathbb{C}$ is the irreducible representation with highest weight λ of the complex reductive group $G \times \mathbb{C}$ and for every field k, $V(\lambda) \otimes k$ is the Weyl module of $G \times k$ mentioned above, see [22, II.8.3] or [34, p. 212]. Consequently, the question in the preceding paragraph is the same as asking: *For which G and λ is it true that $V(\lambda) \otimes k$ is an irreducible representation of $G \times k$ for every field k?* Because G is split, $V(\lambda) \otimes k$ is irreducible if and only if $V(\lambda) \otimes P$ is irreducible where P is the prime field of k.[4] Therefore, it is natural to call such $V(\lambda)$ *globally irreducible*.

There is a well known and elementary sufficient criterion:

$$\textit{If } \lambda \textit{ is minuscule, then } V(\lambda) \otimes k \textit{ is irreducible for every field } k. \tag{1}$$

See §2 for the definition of minuscule. This provides an important family of examples, because representations occurring in this way include $\Lambda^r(V)$ for $1 \leq r < n$ where V is

[3] The definitions of these three modules make sense also when $\mathrm{char}\, k = 0$, and in that case all four modules are isomorphic.

[4] See [22, II.2.9]. For a detailed study of how this fails when G is not split, see [37].