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FINITENESS PROPERTIES OF THE JOHNSON SUBGROUPS

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ABSTRACT. The main goal of this note is to provide evidence that the first rational homology of the Johnson subgroup $K_{g,1}$ of the mapping class group of a genus g surface with one marked point is finite-dimensional. Building on work of Dimca-Papadima [4], we use symplectic representation theory to show that, for all $g > 3$, the completion of $H_1(K_{g,1}, \mathbb{Q})$ with respect to the augmentation ideal in the rational group algebra of $\mathbb{Z}^{2g}$ is finite-dimensional. We also show that the terms of the Johnson filtration of the mapping class group have infinite-dimensional rational homology in some degrees in almost all genera, generalizing a result of Akita.

1. INTRODUCTION

Let π denote the fundamental group of a closed orientable surface S_g of genus $g \geq 2$ and let $\Gamma_{g,r}$ denote the mapping class group of S_g with r marked points, where $r = 0$ or 1.

The *nth Johnson subgroup* $K_{g,1}(n)$ of $\Gamma_{g,1}$ and the *nth outer Johnson subgroup* $K_g(n)$ of $\Gamma_g := \Gamma_{g,0}$ are defined, respectively, to be the kernels of the natural maps

$$\Gamma_{g,1} \to \mathrm{Aut}(\pi/\pi^{(n+1)}) \qquad \Gamma_g \to \mathrm{Out}(\pi/\pi^{(n+1)}).$$

Here $\pi^{(n)}$ denotes the nth term of the lower central series of π. The classical Torelli groups $T_{g,1}$ and T_g are recovered by taking $n = 1$, and the original Johnson subgroups $K_{g,1}$ and K_g are recovered by taking $n = 2$.

The finiteness properties of the Johnson subgroups are poorly understood. For example, it is not known whether these are finitely generated for even a single $n \geq 2$. In [4], Dimca-Papadima showed that $H_1(K_g, \mathbb{C})$ is a finite-dimensional vector space as long as $g \geq 4$. It is currently unknown whether $H_1(K_3, \mathbb{Q})$ is finite-dimensional. On the other hand, in [1] Akita showed that $H_\bullet(K_g, \mathbb{Q})$ and $H_\bullet(K_{g,1}, \mathbb{Q})$ are infinite-dimensional when $g \geq 7$, so K_g and $K_{g,1}$ must have some infinite-rank homology in these cases. Mess has shown in [13] that $K_2 = T_2$ is a free group of countably infinite rank, implying that $H_1(K_2, \mathbb{Q})$ is infinite-dimensional. It is currently unknown whether $H_1(K_{g,1}, \mathbb{Q})$ is finite-dimensional when $g \geq 3$.

The main goal of this note is to provide evidence that $H_1(K_{g,1}, \mathbb{Q})$ is finite-dimensional. Recall that if a finite-dimensional k-vector space V is also a module over a commutative noetherian k-algebra A, then the completion of V with respect to any ideal $I \subset A$ is a finite-dimensional k-vector space. Let $H = \pi^{ab}$. Then $H_1(K_{g,1}, \mathbb{Q})$ is a $\mathbb{Q}H$-module[1]. Let J denote the augmentation ideal of $\mathbb{Q}H$.

Theorem 1.1. *For each $g \geq 4$ the J-adic completion $H_1(K_{g,1}, \mathbb{Q})^\wedge$ is finite-dimensional.*

[1]This will be shown in Section 4.