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# Computational aspects of the higher Nash blowup of hypersurfaces

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ABSTRACT

The higher Nash blowup of an algebraic variety replaces singular points with limits of certain spaces carrying higher-order data associated to the variety at non-singular points. In this note we will define a higher-order Jacobian matrix that will allow us to make explicit computations concerning the higher Nash blowup of hypersurfaces. Firstly, we will generalize a known method to compute the fiber of this modification. Secondly, we will give an explicit description of the ideal whose blowup gives the higher Nash blowup. As a consequence, we will deduce a higher-order version of Nobile's theorem for normal hypersurfaces.



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## Introduction

The main purpose of this note is to present, as the title suggests, some computational aspects of the higher Nash blowup of a hypersurface. The higher Nash blowup is defined as follows (see [11,12,15]):

Let $X = \mathbf{V}(I) \subset \mathbb{C}^s$ be an irreducible algebraic variety of dimension $d$, given as the zero set of some ideal $I$. Let $R$ be its ring of regular functions. For each $p \in X$, let $(R_p, \mathfrak{m}_p)$ be the localization at $p$ and define the $R_p/\mathfrak{m}_p \cong \mathbb{C}$-vector space $T^n_p X := (\mathfrak{m}_p/\mathfrak{m}_p^{n+1})^\vee$. This is a vector space of dimension $D = \binom{d+n}{d} - 1$, whenever $p$ is a non-singular point. The fact that $X \subset \mathbb{C}^s$ implies that $T^n_p X \subset T^n_p \mathbb{C}^s \cong \mathbb{C}^E$, where $E = \binom{s+n}{s} - 1$, that is, we can see $T^n_p X$ as an element of the Grassmanian $Gr(D, \mathbb{C}^E)$. Let $S(X)$ be the singular locus of $X$. Now consider the Gauss map:

$$G_n : X \setminus S(X) \to Gr(D, \mathbb{C}^E), \quad p \mapsto T^n_p X.$$

Denote by $Nash_n(X)$ the Zariski closure of the graph of $G_n$. Call $\pi_n$ the restriction to $Nash_n(X)$ of the projection of $X \times Gr(D, \mathbb{C}^E)$ to $X$. When $n = 1$, the pair $(Nash_n(X), \pi_n)$ is usually called the *Nash modification* of $X$. For $n > 1$, $(Nash_n(X), \pi_n)$ is called the *higher Nash blowup* of $X$. This construction gives a canonical modification of an algebraic variety that replaces singular points by limits of sequences $\{T^n_{p_i} X\}$, where $\{p_i\} \subset X$ is any sequence of non-singular points converging to a singular one.

Unfortunately, despite being a natural and geometrically attractive modification, it is hard to compute in general. The goal of this note is to deal with this problem, to some extent, in the case of hypersurfaces. We will start by defining in Section 1 a generalization of the Jacobian matrix that involves also higher-order derivatives, which is more suitable to this context. Using this matrix, we will give in Section 2 some higher-order criteria of non-singularity. Next, we will prove in Section 3 that the spaces $T^n_p X$ can be identified with the kernel of the higher-order Jacobian, as with the tangent space.

In the last section we will give some applications of the previous results. Firstly, we will generalize a method proposed by D. O'Shea which computes limits of tangent spaces to a singular point of a hypersurface (see [13]). This method, along with the theory of Gröbner bases, will allow us to compute examples showing some interesting phenomena of the set of limits of spaces $T^n_p X$. Later, using some results of O. Villamayor appearing in [14], we will explicitly describe the ideal whose blowup gives the higher Nash blowup by means of the higher-order Jacobian matrix.

In [15], the author proposes the following conjecture: given an algebraic variety $X$, $Nash_n(X)$ is non-singular for $n \gg 0$. If this conjecture is true, the higher Nash blowup would give a canonical resolution of singularities in one step. Implicit in this question is the fact that the higher Nash blowup is a non-trivial modification of a variety. For $n = 1$ this fact is known as Nobile's theorem: the Nash modification of a variety is an isomorphism if and only if the variety is non-singular (see [11]). We will prove the