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Gotzmann's persistence theorem for finite modules

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ABSTRACT

We prove a generalization of Gotzmann's persistence theorem in the case of modules with constant Hilbert polynomial. As a consequence, we show that the defining equations that give the embedding of a Quot scheme of points into a Grassmannian are given by a single Fitting ideal.

0. Introduction

One of the main results of this paper is the following.

Theorem A. *Let A be a ring, let $S = A[X_0, \dots, X_r]$, let $M = \bigoplus_{i=1}^{p} S$, and let N be a graded S-submodule of M, generated in degrees at most d. Write $Q = M/N$ and let $n \leq d$. If Q_t is locally free of rank n for $t = d$ and $t = d + 1$, then Q_t is locally free of rank n for all $t \geq d$.*

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This theorem concerns homogeneous submodules $N \subseteq \bigoplus_{i=1}^{p} A[X_0, \ldots, X_r]$ generated in degrees at most d, for some d. A special case of such a submodule is a homogeneous ideal $I \subseteq A[X_0, \ldots, X_r]$ generated in degrees at most d. In that case, when A is noetherian, we have Gotzmann's persistence theorem [7] which states that if the graded component Q_t of the quotient $Q = A[X_0, \ldots, X_r]/I$ is flat over A for $t = d$ and $t = d+1$, and $\operatorname{rank}_A Q_{d+1} = (\operatorname{rank}_A Q_d)^{\langle d \rangle}$, then there are two implications. Firstly, the graded component Q_t is flat over A for all $t \geq d$, so Q has a Hilbert polynomial $P(t)$. Secondly, the theorem states that $P(t+1) = \operatorname{rank}_A Q_{t+1} = (\operatorname{rank}_A Q_t)^{\langle t \rangle}$ for all $t \geq d$.

We have here used Macaulay representations to describe the assumption on the rank in Gotzmann's persistence theorem, see [1, Section 4.2]. In other words, the theorem says that if the Hilbert function of Q has a maximal growth from degree d to degree $d+1$, then the homogeneous components in all higher degrees are flat and the Hilbert function $h(t) = \operatorname{rank}_A Q_t$ equals the Hilbert polynomial $P(t)$ for all degrees $t \geq d$. In the case when $\operatorname{rank}_A Q_d = n \leq d$, we have that $(\operatorname{rank}_A Q_t)^{\langle t \rangle} = n$ for all $t \geq d$. Thus, Theorem A is a generalization of this result for the case with constant Hilbert polynomial $P(t) = n$.

Note that Gasharov [5] has proved a generalization of Gotzmann's persistence theorem to modules in the case of polynomial rings over fields, where the flatness is trivial, and that our result extends this to polynomial rings over arbitrary rings.

Our interest in the result of Theorem A comes from its application to Quot schemes. In algebraic geometry, Gotzmann's persistence theorem has been used to find defining equations of Hilbert schemes, see, e.g., [7] and [10, Appendix C]. The Quot scheme is a generalization of both Hilbert schemes and Grassmannians, and is therefore a natural object to study. It was first introduced by Grothendieck who also proved its existence using an embedding into a Grassmannian [8]. This embedding was however only given abstractly. For the case with constant Hilbert polynomials, Skjelnes proved that the embedding of the Quot scheme of points into a Grassmannian is given by an infinite intersection of closed subschemes defined by certain Fitting ideals [14]. Moreover, Skjelnes mentions that proving a generalization of Gotzmann's persistence theorem to modules would also prove that only one of those closed subschemes suffices to describe the embedding. We make this statement precise by showing the following consequence of Theorem A.

Theorem B. *Let V be a projective and finitely generated module over a noetherian ring A. Let $\mathcal{O}_{\mathbb{P}(V)}^{\oplus p}$ denote the free sheaf of rank p on $f \colon \mathbb{P}(V) \to \operatorname{Spec}(A) = S$. Fix two integers $n \leq d$, and let $g \colon G \to S$ denote the Grassmannian scheme parametrizing locally free rank n quotients of $f_* \mathcal{O}_{\mathbb{P}(V)}^{\oplus p}(d)$. We let*

$$0 \longrightarrow \mathcal{R}_d \longrightarrow g^* f_* \mathcal{O}_{\mathbb{P}(V)}^{\oplus p}(d) \longrightarrow \mathcal{E}_d \longrightarrow 0$$

denote the universal short exact sequence on the Grassmannian G, and let $\mathcal{E}_{d+1}$ be the cokernel of the induced map $\mathcal{R}_d \otimes_{\mathcal{O}_G} g^* f_* \mathcal{O}_{\mathbb{P}(V)}(1) \to g^* f_* \mathcal{O}_{\mathbb{P}(V)}^{\oplus p}(d+1)$. Then, we have that