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Group gradings on the Lie algebra of upper triangular matrices

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The algebras UT_n of the $n \times n$ upper triangular matrices over a field K are of significant importance in the theory of algebras with polynomial identities. Group gradings on algebras appear in various areas and provide an indispensable tool in the study of the algebraic and combinatorial properties of the algebras in question. We classify the group gradings on the Lie algebra $UT_n^{(-)}$. It was proved by Valenti and Zaicev in 2007 that every group grading on the associative algebra UT_n is isomorphic to an elementary grading. The elementary gradings on UT_n are also well understood, see [6]. It follows from our results that there are nonelementary gradings on $UT_n^{(-)}$. Thus the gradings on the Lie algebra $UT_n^{(-)}$ are much more intricate than those in the associative case.

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Introduction

Group gradings in Commutative algebra are a natural generalization of properties of polynomial rings. Concerning the noncommutative case, Wall [13] described the finite dimensional graded simple algebras when the grading group is $\mathbb{Z}_2$, the cyclic group of order 2. Later on gradings on algebras became an object of extensive study. One may consult the papers [1,2], and the recent monograph [9] and the bibliography therein for further, more detailed information concerning gradings on algebras. We recall some important results in the area that will be needed here.

Fix a group G, a field K, and a K-algebra A. Here we do not require A to be associative or commutative. The algebra A is G-graded if $A = \oplus_{g\in G} A_g$, a direct sum of vector subspaces such that $A_gA_h \subseteq A_{gh}$ for all g, $h \in G$. A vector subspace V of A is graded (also called homogeneous) if $V = \oplus_{g\in G} V \cap A_g$, analogously for graded ideals and subalgebras of A. If A and B are G-graded algebras, a homomorphism $f\colon A \to B$ is a graded homomorphism if it respects the grading, that is $f(A_g) \subseteq B_g$ for all $g \in G$.

The gradings on simple finite dimensional algebras are quite well understood. For example the classification of the gradings on the (associative) matrix algebras of order n was obtained by Bahturin and Zaicev, see [3]. Later on their description was extended, in the case of abelian groups, to simple associative algebras of arbitrary dimension, over an algebraically closed field, provided they possess a minimal one sided ideal, see [9, pp. 27, 28]. Similar results were obtained in the case of simple finite dimensional Lie algebras, see for example [9] for an extensive collection of the state-of-art. The analogous problem for simple Jordan algebras was dealt with as well, see [4] and its bibliography.

On the other hand relatively little is known about the gradings on important algebras that fail to be simple. The Grassmann (or exterior) algebra appears naturally in various branches of Mathematics and Physics; the gradings on this algebra are known in several limited cases, see for example [7]. The gradings on the associative algebra of the upper triangular matrices of any order were classified by Valenti and Zaicev in [12]. The gradings on the Jordan algebra of the upper triangular matrices of order 2 were described in [10]. Note that the latter algebra is a Jordan algebra of a symmetric bilinear form though the form is degenerate. Recently Bahturin studied gradings on nilpotent algebras that are free in a variety of algebras [5].

The elementary gradings (also called good gradings) appear in the classification of the gradings on matrix algebras over a field. A grading on $M_n(K)$ is elementary whenever the matrix units e_{ij} are all homogeneous in the grading. There are other, equivalent definitions of an elementary grading, see below. The notion of an elementary grading can be transferred in a natural manner to subalgebras and vector subspaces of $M_n(K)$ spanned by matrix units. One can find additional information about gradings on matrix algebras in the papers [2], [1], and also in the recent monograph [9].

The elementary gradings on the associative algebra UT_n of the upper triangular matrices of order n over a field K were described in [6]. Also, in [12], the authors proved that any grading on UT_n is, up to a graded isomorphism, elementary. Hence the results