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Vanishing theorems for linearly obstructed divisors

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A B S T R A C T

We study divisors on the blow-up of $\mathbb{P}^n$ at points in general position that are non-special with respect to the notion of linear speciality introduced in [6]. We describe the cohomology groups of their strict transforms via the blow-up of the space along their linear base locus. We extend the result to non-effective divisors that sit in a small region outside the effective cone. As an application, we describe linear systems of divisors in $\mathbb{P}^n$ blown-up at points in star configuration and their strict transforms via the blow-up of the linear base locus.

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1. Introduction

The motivation for studying vanishing theorems of divisors comes from Birational Geometry (Mori's Minimal Model Program, see [19]) and Commutative Algebra (higher order embeddings of projective varieties, see [2]). In particular, vanishing theorems have applications to positivity properties of divisors such as global generation, very ampleness and, more in general, k-very ampleness properties.

We denote by $\mathcal{L} = \mathcal{L}_{n,d}(m_1, \ldots, m_s)$ the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ passing through a collection of s points in general position with multiplicities at least $m_1, \ldots, m_s \geq 0$ respectively. The *(affine) virtual dimension* of $\mathcal{L}$ is denoted by

$$\mathrm{vdim}(\mathcal{L}) = \binom{n+d}{n} - \sum_{i=1}^{s} \binom{n+m_i-1}{n},$$

and the *expected dimension* of $\mathcal{L}$ is defined to be $\mathrm{edim}(\mathcal{L}) = \max(\mathrm{vdim}(\mathcal{L}), 0)$. The problem of computing the dimension of such linear systems is often referred to as *(polynomial) interpolation problem in* $\mathbb{P}^n$ (see e.g. [9] for an account).

If D is the strict transform of a general divisor in $\mathcal{L}$ in the blow-up X of $\mathbb{P}^n$ at the s points,

$$D := dH - \sum_{i=1}^{s} m_i E_i, \tag{1.1}$$

then $\mathrm{vdim}(D) := \mathrm{vdim}(\mathcal{L})$ equals $\chi(X, \mathcal{O}_X(D))$, the Euler characteristic of the sheaf on X associated with D, while $\dim(\mathcal{L})$ is the number of global section of $\mathcal{O}_X(D)$, namely the dimension of the space $H^0(X, \mathcal{O}_X(D))$. Using the terminology of the interpolation problem, we will refer to D as a divisor of degree d *interpolating* s general points with assigned multiplicities $m_1, \ldots, m_s$.

The inequality $\dim(\mathcal{L}) \geq \mathrm{edim}(\mathcal{L})$ is always satisfied. However, if the conditions imposed by the assigned multiple points are not linearly independent, then the actual dimension of $\mathcal{L}$ is strictly greater than the expected one: in that case we say that $\mathcal{L}$ (or D) is *special*. Otherwise, whenever the actual and the expected dimension coincide we say that $\mathcal{L}$ is *non-special*. The *speciality* of $\mathcal{L}$ (or D) is defined to be the difference $\dim(\mathcal{L}) - \mathrm{edim}(\mathcal{L})$.

In the last century the problem of computing the dimension (or, equivalently, computing the speciality) of linear systems was studied with different techniques by many people. In the planar case, the Segre–Harbourne–Gimigliano–Hirschowitz conjectures predicts all special linear systems. This famous conjecture gives information about the Mori cone of X, $\overline{\mathrm{NE}}(X)$, together with its dual, the nef cone of X. For example, one of its implications is the so called (-1)-Curves Conjecture, see [10, Conjecture 3.2.1] and [11, Conjecture 1.1]. This consists of a geometric description of the Mori cone of X: while the K-negative part of $\overline{\mathrm{NE}}(X)$, namely the set of classes intersecting negatively