Journal of Algebra 477 (2017) 390–434

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Fundamental invariants of orbit closures

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A R T I C L E   I N F O

Article history:
Received 30 May 2016
Available online 16 January 2017
Communicated by Shrawan Kumar

MSC:
68Q17
13A50
14L24

Keywords:
Geometric complexity theory
Invariants
Orbit closure
Non-normality
Alon–Tarsi conjecture

A B S T R A C T

For several objects of interest in geometric complexity theory, namely for the determinant, the permanent, the product of variables, the power sum, the unit tensor, and the matrix multiplication tensor, we introduce and study a fundamental SL-invariant function that relates the coordinate ring of the orbit with the coordinate ring of its closure. For the power sums we can write down this fundamental invariant explicitly in most cases. Our constructions generalize the two Aronhold invariants on ternary cubics. For the other objects we identify the invariant function conditional on intriguing combinatorial problems much like the well-known Alon–Tarsi conjecture on Latin squares. We provide computer calculations in small dimensions for these cases. As a main tool for our analysis, we determine the stabilizers, and we establish the polystability of all the mentioned forms and tensors (including the generic ones).

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[1] Partially supported by DFG grant BU 1371/3-2.
[2] This research was conducted while CI was at Texas A&M University.

http://dx.doi.org/10.1016/j.jalgebra.2016.12.035

1. Introduction

1.1. Motivation

In 1979 Valiant showed that for every polynomial p there exists a matrix A whose entries are affine linear forms in the variables of p such that $\det(A) = p$. The minimal size of such a matrix is called the *determinantal complexity* $\mathsf{dc}(p)$. The size $\mathsf{dc}(p)$ is closely related to the number of computation gates in arithmetic circuits computing p, see [53,52,32] for more information. Valiant's flagship conjecture is concerned with the determinantal complexity of the permanent polynomial:

$$\mathrm{per}_m := \sum_{\pi \in S_m} X_{1,\pi(1)} X_{2,\pi(2)} \cdots X_{m,\pi(m)},$$

a homogeneous degree m polynomial in m^2 variables.

Conjecture 1.1. The sequence $\mathsf{dc}(\mathrm{per}_m)$ grows superpolynomially fast.

To study this conjecture, Mulmuley and Sohoni [40,41] proposed an approach through algebraic geometry and representation theory, for which they coined the term *geometric complexity theory*. The main idea is to study the GL_{n^2}-orbit closure $\overline{\mathrm{GL}_{n^2} \det_n}$ of the determinant polynomial

$$\det_n := \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) X_{1,\pi(1)} X_{2,\pi(2)} \cdots X_{n,\pi(n)}$$

in the space $\mathsf{Sym}^n \mathbb{C}^{n^2}$ of all homogeneous degree n polynomials in n^2 variables. For $n > m$ we compare this closure to the GL_{n^2}-orbit closure $\overline{\mathrm{GL}_{n^2} \mathrm{per}_{n,m}}$ of the *padded* permanent $\mathrm{per}_{n,m} := (X_{n,n})^{n-m} \mathrm{per}_m$. We consider the GL_{n^2}-action on the coordinate rings of these closures and note the following: If there exists an irreducible representation in $\mathcal{O}(\overline{\mathrm{GL}_{n^2} \mathrm{per}_{n,m}})$ that does not lie in $\mathcal{O}(\overline{\mathrm{GL}_{n^2} \det_n})$, then $\mathsf{dc}(\mathrm{per}_m) > n$.

A main approach towards understanding $\mathcal{O}(\overline{\mathrm{GL}_{n^2} \det_n})$ is by studying it as a subalgebra of the coordinate ring $\mathcal{O}(\mathrm{GL}_{n^2} \det_n)$ of the orbit $\mathrm{GL}_{n^2} \det_n$. The representation theoretic decomposition of $\mathcal{O}(\mathrm{GL}_{n^2} \det_n)$ can be deduced rather explicitly using the algebraic Peter–Weyl theorem [8, §4 and §5]. The main motivation of this paper is to improve our understanding of the connection between the coordinate ring of the orbit $\mathrm{GL}_{n^2} \det_n$ and its closure $\overline{\mathrm{GL}_{n^2} \det_n}$.

1.2. Results

We consider the space $\mathsf{Sym}^D \mathbb{C}^m$ of (homogeneous) forms of degree D in m variables with the natural action of the group GL_m. Let $w \in \mathsf{Sym}^D \mathbb{C}^m$ be a polystable form, which means that its SL_m-orbit is closed. We define the *stabilizer period* $a(w)$ as the