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Computing the signatures of subgroups of non-Euclidean crystallographic groups

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ABSTRACT

A (planar and cocompact) non-Euclidean crystallographic (NEC) group Δ is a subgroup of the group of, conformal and anti-conformal, isometries of the hyperbolic plane $\mathbb{H}^2$ such that $\mathbb{H}^2/\Delta$ is compact. NEC groups are classified algebraically by a symbol called signature. In this symbol there is a sign $+$ or $-$ and, in the case of sign $+$, some cycles of integers in the signature, called period-cycles, have an essential direction. In 1990 A.H.M. Hoare gave an algorithm to obtain the signature of a finite index subgroup of an NEC group. The process of Hoare fails in some cases in the task of computing the direction of period-cycles. In this work we complete the algorithm of Hoare, this allows us to construct a program for computing the signature of subgroups of NEC groups in all cases.

1. Introduction

A (planar) non-Euclidean crystallographic group Δ is a discrete subgroup of the group of (conformal and anti-conformal) isometries of the hyperbolic plane $\mathbb{H}^2$. We shall con-

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sider only cocompact NEC groups, i.e. we assume that $\mathbb{H}^2/\Delta$ is compact. The algebraic structure of Δ is given by a symbol called signature (see [16,13]):

$$(g;\sigma;[m_1,...,m_r];\{(n_{11},...,n_{1s_1}),...,(n_{k1},...,n_{ks_k})\})$$

where g is the genus of the surface $\mathbb{H}^2/\Delta$, $\sigma = +$ or $-$ is the orientability character of $\mathbb{H}^2/\Delta$, $[m_1,...,m_r]$ is the set of branched indices (periods) of the covering $\mathbb{H}^2 \to \mathbb{H}^2/\Delta$ with values in interior points of $\mathbb{H}^2/\Delta$, one period for each branched value, and the ordered sets (period-cycles) of branched indices: $(n_{11},...,n_{1s_1}),...,(n_{k1},...,n_{ks_k})$, correspond to branched values in the k boundary components of $\mathbb{H}^2/\Delta$.

Two signatures:

$$(g;\sigma;[m_1,...,m_r];\{(n_{11},...,n_{1s_1}),...,(n_{k1},...,n_{ks_k})\})$$
$$(g';\sigma';[m'_1,...,m'_r];\{(n'_{11},...,n'_{1s_1}),...,(n'_{k1},...,n'_{ks_k})\})$$

are considered equivalent if:

1. $g = g'$; $\sigma = \sigma'$;
2. $r = r'$; $(m'_1,...,m'_r) = (m'_{\varepsilon(1)},...,m'_{\varepsilon(r)})$, $\varepsilon \in \Sigma\{1,...,r\}$;
3. $k = k'$; $s_i = s'_{\delta(i)}$, $\delta \in \Sigma\{1,...,k\}$;
4. Let $\alpha = (1,...,s_i) \in \Sigma\{1,...,s_i\}$

- If $\sigma = \sigma' = +$, either:

a. For all i, $(n_{i1},...,n_{is_i}) = (n'_{\delta(i)\theta_i(1)},...,n'_{\delta(i)\theta_i(s_i)})$, $\theta_i = \alpha^{l_i}$, $0 \le l_i \le s_i$, or
b. For all i, $(n_{is_i},...,n_{i1}) = (n'_{\delta(i)\theta_i(1)},...,n'_{\delta(i)\theta_i(s_i)})$, $\theta_i = \alpha^{l_i}$, $0 \le l_i \le s_i$.

- If $\sigma = \sigma' = -$, for each i, $i = 1,...,k$, either:

a. $(n_{i1},...,n_{is_i}) = (n'_{\delta(i)\theta_i(1)},...,n'_{\delta(i)\theta_i(s_i)})$, $\theta_i = \alpha^{l_i}$ or
b. $(n_{is_i},...,n_{i1}) = (n'_{\delta(i)\theta_i(1)},...,n'_{\delta(i)\theta_i(s_i)})$, $\theta_i = \alpha^{l_i}$.

Following the terminology in [13]: in the orientable case ($\sigma = +$) corresponding pairs of period-cycles are all paired in the same way, all directly or all inversely. In the non-orientable case, some are paired directly and some inversely.

Two NEC groups are isomorphic if and only if they have equivalent signatures. Each NEC admits a canonical presentation, the geometrical type of the generators and the word expressions of relations of a canonical presentation is given by the signature.

If Γ is a finite index subgroup of Δ then Γ is also an NEC group. From a presentation of Δ and the action of the generators of such presentation on the cosets Δ/Γ, the Reidemeister–Schreier method, provides a non-canonical presentation of Γ. But to obtain the signature of Γ from a non-canonical presentation is not an easy task.

If $\Delta \le \mathrm{Isom}^+(\mathbb{H}^2)$ then we say that Δ is a Fuchsian group and there is a direct method of Singerman [14] to obtain the signature of a subgroup Γ of Δ. This method