Journal of Algebra 477 (2017) 496–515

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A theory of pictures for quasi-posets

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A R T I C L E I N F O

Article history:
Received 29 November 2016
Communicated by Jean-Yves Thibon

Keywords:
Combinatorial Hopf algebra
Pictures
Quasi-poset
Finite topology

A B S T R A C T

The theory of pictures between posets is known to encode much of the combinatorics of symmetric group representations and related topics such as Young diagrams and tableaux. Many reasons, combinatorial (e.g. since semi-standard tableaux can be viewed as double quasi-posets) and topological (quasi-posets identify with finite topologies) lead to extend the theory to quasi-posets. This is the object of the present article.

Introduction

The theory of pictures between posets is known to encode much of the combinatorics of symmetric group representations and related topics such as preorder diagrams and tableaux. The theory captures for example the Robinson–Schensted (RS) correspondence

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http://dx.doi.org/10.1016/j.jalgebra.2017.01.003

or the Littlewood–Richardson formula, as already shown by Zelevinsky in the seminal article [20]. Recently, the theory was extended to double posets (pairs of orders coexisting on a given finite set – hereafter, "order" means "partial order"; an order on X defines a poset structure on X) and developed from the point of view of combinatorial Hopf algebras which led to new advances in the field [16,8–10].

In applications, a fundamental property that has not been featured enough, is that often pictures carry themselves implicitly a double poset structure. A typical example is given by standard Young tableaux, which can be put in bijection with certain pictures (this is one of the nicest ways in which their appearance in the RS correspondence can be explained [20]) and carry simultaneously a poset structure (induced by their embeddings into $\mathbb{N} \times \mathbb{N}$ equipped with the coordinate-wise partial order) and a total order (the one induced by the integer labelling of the entries of the tableaux).

However, objects such as tableaux with repeated entries, such as semi-standard tableaux, although essential, do not fit into this framework. They should actually be thought of instead as double quasi-posets (pairs of preorders on a given finite set): the first preorder is the same than for standard tableaux (it is an order), but the labelling by (possibly repeated) integers is naturally captured by a preorder on the entries of the tableau (the one for which two entries are equivalent if they have the same label and else are ordered according to their labels).

Besides the fact that these ideas lead naturally to new results and structures on preorders, other observations and motivations have led us to develop on systematic bases in the present article a theory of pictures for quasi-posets. Let us point out in particular recent developments (motivated by applications to multiple zeta values, Rota–Baxter algebras, stochastic integrals... [4,2,3]) that extend to surjections [18,17,14,13] the theory of combinatorial Hopf algebra structures on permutations [15,7]. New results on surjections will be obtained in the last section of the article.

Lastly, let us mention our previous works on finite topologies (equivalent to quasi-posets) [11,12] (see also [5,6] for recent developments) which featured the two products defined on finite topologies by disjoint union and the topological join product. The same two products, used simultaneously, happen to be the ones that define on double quasi-posets an algebra (and actually self-dual Hopf algebra) structure extending the usual one on double posets.

The article is organized as follows. Section 1 introduces double quasi-posets. Sections 2 and 3 introduce and study Hopf algebra structures on double quasi-posets. Section 4 defines pictures between double quasi-posets. Due to the existence of equivalent elements for both preorders of a double quasi-poset, the very notion of pictures is much more flexible than for double posets. From Section 5 onwards, we focus on the algebraic structures underlying the theory of pictures for double quasi-posets. Section 5 investigates duality phenomena and shows that pictures define a symmetric Hopf pairing on the Hopf algebra of double quasi-posets. Section 6 addresses the question of internal products, generalizing the corresponding results on double posets. Internal products (by which we mean the existence of an associative product of double posets within a given cardinality)