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Separated monic representations I: Gorenstein-projective modules

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A B S T R A C T

For a finite acyclic quiver Q, an ideal I of a path algebra kQ generated by monomial relations, and a finite-dimensional k-algebra A, we introduce the separated monic representations of a bound quiver (Q, I) over A. They differ from the (usual) monic representations. The category $\operatorname{smon}(Q, I, A)$ of the separated monic representations of (Q, I) over A coincides with the category $\operatorname{mon}(Q, I, A)$ of the (usual) monic representations if and only if $I = 0$ and each vertex of Q is the ending vertex of at most one arrow. We give properties of the structural maps of separated monic representations, and prove that $\operatorname{smon}(Q, I, A)$ is a resolving subcategory of $\operatorname{rep}(Q, I, A)$. We introduce the condition (G). Let $\Lambda := A \otimes kQ/I$. By the equivalence $\operatorname{rep}(Q, I, A) \cong \Lambda\text{-mod}$ of categories, the main result claims that a Λ-module is Gorenstein-projective if and only if it is in $\operatorname{smon}(Q, I, A)$ and has a local A-Gorenstein-projective property (G). As consequences, the separated monic Λ-modules are exactly the projective Λ-modules if and only if A is semi-simple; and they are exactly the Gorenstein-projective Λ-modules if and only if A is self-injective, and if and only if $\operatorname{smon}(Q, I, A)$ is a Frobenius category.

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Introduction

Throughout k is a field, A a finite-dimensional k-algebra, Q a finite acyclic quiver, I an ideal of a path algebra kQ generated by monomial relations, and

$$\Lambda = A \otimes \; kQ/I.$$

All tensors are over k if not specified otherwise, and all modules considered are finitely generated. Denote by A-mod the category of left A-modules. We study the representation category $\mathrm{rep}(Q, I, A)$ for the bound quiver (Q, I) over A instead of the representation category $\mathrm{rep}(Q, I, k)$ of (Q, I) over k ([21]), or equivalently, the module category Λ-mod. When Q is of type A_2, it is exactly the morphism category of A ([2]).

Our aim is to describe the Gorenstein-projective Λ-modules ([3,18]). They enjoy more stable properties than the projective modules (see e.g. [11,42,25,28]) and are widely used in the relative homological algebra, representation theory and algebraic geometry (see e.g. [4,8,16,12,17]). A striking feature is that the category $\mathcal{GP}(A)$ of Gorenstein-projective A-modules is a Frobenius category (see e.g. [9, Prop. 3.8]), and hence its stable category $\underline{\mathcal{GP}}(A)$ is triangulated ([22]). If A is a Gorenstein algebra, then $\mathcal{GP}(A)$ has Auslander–Reiten sequences, and hence $\underline{\mathcal{GP}}(A)$ has Auslander–Reiten triangles; also in this case $\underline{\mathcal{GP}}(A)$ is just the singularity category of A ([11,23,34]), up to a triangle-equivalence. Note that if A is Gorenstein then so is $\Lambda = A \otimes kQ/I$ ([5, Prop. 2.2]), and hence $\mathcal{GP}(\Lambda)$ has Auslander–Reiten sequences, and $\underline{\mathcal{GP}}(\Lambda)$ has Auslander–Reiten triangles.

For this aim, we introduce the *separated monic representations* of (Q, I) over A, or equivalently, the *separated monic* Λ-modules, and study *the separated monomorphism category* $\mathrm{smon}(Q, I, A)$ of separated monic representations of (Q, I) over A (see Definition 2.1). It relates with but differs from the (usual) monomorphism category $\mathrm{mon}(Q, I, A)$ (Subsection 2.2). In fact, $\mathrm{smon}(Q, I, A) = \mathrm{mon}(Q, I, A)$ if and only if $I = 0$ and each vertex of Q is the ending vertex of at most one arrow (Proposition 2.3). If $I = 0$, $\mathrm{smon}(Q, A) := \mathrm{smon}(Q, 0, A)$ has been studied in [32]. In the special case of Q being of type A_n with linear orientation and $I = 0$, this category $\mathrm{smon}(Q, A) = \mathrm{mon}(Q, A) := \mathrm{mon}(Q, 0, A)$ strongly relates with the representations of a chain (as a partially ordered set) over A ([43–45]); and it is a Frobenius category when A is a self-injective algebra. However, in general $\mathrm{mon}(Q, I, A)$ is no longer Frobenius when A is self-injective; while $\mathrm{smon}(Q, I, A)$ has the advantage that it is always Frobenius when A is self-injective.

In fact, initiated by G. Birkhoff [10], there is a long history of studying $\mathrm{smon}(Q, A)$, where Q is of type A_n with linear orientation. It is called *the submodule category* for $n = 2$ in [38–40], and *the filtered chain category* in [44–46]. C.M. Ringel and M. Schmidmeier have established the Auslander–Reiten theory of the submodule category ([39]; also [47]); and D. Simson has studied the representation type ([44,45]). A reciprocity of the monomorphism operator and the perpendicular operator is given ([48]). According