Journal of Algebra 479 (2017) 78–101

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# The relation $\underset{LR}{\leqslant}$ on some elements of the affine Weyl group $\widetilde{C}_n$ ☆

Jian-yi Shi

*Department of Mathematics, East China Normal University, Shanghai, 200241, PR China*

ARTICLE INFO

*Article history:*
Received 24 June 2016
Available online 3 February 2017
Communicated by Shrawan Kumar

*Keywords:*
Affine Weyl group
The longest element
Left cell
Two-sided cell
Partition
Iterating star operations

ABSTRACT

Let $(W, S)$ be the affine Weyl group of type $\widetilde{C}_n$ with $S$ its Coxeter generator set. Let $\overline{\Lambda}_{2n+1}$ be the set of all partitions $\lambda = (\lambda_1, ..., \lambda_r)$ of $2n+1$ such that $\sum_{j=1}^{2k+1} \lambda_j$ is odd for any $k \in \mathbb{N}$ with $2k+1 \leqslant r$. For any $J \subsetneq S$, let $w_J$ be the longest element in the parabolic subgroup of $W$ generated by $J$. We define a map $\overline{\phi} : \{w_J \mid J \subsetneq S\} \longrightarrow \overline{\Lambda}_{2n+1}$ and study the preorder $\underset{LR}{\leqslant}$ on the set $\{w_J \mid J \subsetneq S\}$ and its relation with the partial order $\leqslant$ on the set $\{\overline{\phi}(w_J) \mid J \subsetneq S\}$, where iterating star operations and primitive pairs play an important role.



## 0. Introduction

In order to construct representations of a Coxeter group $(W, S)$ and the associated Hecke algebra $\mathcal{H}(W)$, Kazhdan and Lusztig introduced certain preorder relation $\underset{LR}{\leqslant}$ and the corresponding equivalence relation $\underset{LR}{\sim}$ in $W$. The equivalence classes of $W$ with

☆ Supported by the NSFC (11131001 and 11471115), Shanghai Key Laboratory of PMMP, and STCSM (13dz2260400).

*E-mail address:* jyshi11@yahoo.com.

http://dx.doi.org/10.1016/j.jalgebra.2017.01.024
0021-8693/

respect to $\underset{\mathrm{LR}}{\sim}$ are called two-sided cells and those cells provide representations of $W$ and $\mathcal{H}(W)$ with certain good structural properties (see [2]). It is desirable (but difficult in general) to determine, for any $y, w \in W$, whether or not the relation $y \underset{\mathrm{LR}}{\leqslant} w$ holds. In the present paper, we consider the case where $W$ is the affine Weyl group $\widetilde{C}_n$ for any $n \geqslant 2$ and $y, w \in \{w_J \mid J \subsetneq S\}$, where $w_J$ is the longest element in the subgroup $W_J$ of $\widetilde{C}_n$ generated by $J$. Let $\overline{\Lambda}_{2n+1}$ be the set of all partitions $(\lambda_1, \lambda_2, ..., \lambda_r)$ of $2n+1$ for some $r \geqslant 1$ such that $\sum_{i=1}^{2k+1} \lambda_i$ is odd for every $k \in \mathbb{N}$. By Proposition 2.2, we define a map $\overline{\phi} : \{w_J \mid J \subsetneq S\} \longrightarrow \overline{\Lambda}_{2n+1}$. Then we study the preorder $\underset{\mathrm{LR}}{\leqslant}$ on the set $\{w_J \mid J \subsetneq S\}$ and its relation with the partial order $\leqslant$ on the set $\{\overline{\phi}(w_J) \mid J \subsetneq S\}$. The main results about these are Theorems 4.8–4.9 and Propositions 5.8–5.9. The crucial technical tools for our purpose are the iterating star operations on the elements of $\widetilde{C}_n$ and the left primitive pairs, where the former are a certain generalization to $\widetilde{C}_n$ of those introduced by the author in the case of the affine Weyl group $\widetilde{A}_l$, $l \geqslant 1$ (see [6, Chapter 8]), and the latter was introduced in [7, Subsection 3.3].

Let Cell $(\widetilde{C}_n)$ be the set of all two-sided cells of $\widetilde{C}_n$. We define a map $\overline{\psi} : \widetilde{C}_n \longrightarrow \overline{\Lambda}_{2n+1}$ in 5.3 and propose a conjecture (i.e., Conjecture 5.4) that $\overline{\psi}$ induces an order-reversing bijection from the poset (Cell $(\widetilde{C}_n), \underset{\mathrm{LR}}{\leqslant}$) to $(\overline{\Lambda}_{2n+1}, \leqslant)$.

The contents are organized as follows. We collect some concepts, notation and known results in Section 1. Then in Section 2, we define the map $\overline{\phi} : \{w_J \mid J \subsetneq S\} \longrightarrow \overline{\Lambda}_{2n+1}$. We introduce the iterating star operations on elements of $\widetilde{C}_n$ in Section 3. Then we describe the relation $\underset{\mathrm{LR}}{\leqslant}$ on the set $\{w_J \mid J \subsetneq S\}$ in Sections 4–5.

## 1. Preliminaries

In the present section, we collect some concepts, notation and known results for later use.

**1.1.** Denote by $\mathbb{Z}$ (respectively, $\mathbb{N}$, $\mathbb{P}$) the set of all integers (respectively, non-negative integers, positive integers). Denote $[k, n] := \{k, k+1, ..., n\}$ and $[m] := [1, m]$ for any $k \leqslant n$ in $\mathbb{Z}$ and $m \in \mathbb{P}$. For a Coxeter group $W = (W, S)$ with $S$ its Coxeter generator set, let $\leqslant$ be the Bruhat–Chevalley order and $\ell(w)$ the length function on $W$. Let $\mathcal{A} = \mathbb{Z}[u, u^{-1}]$ be the ring of all Laurent polynomials in an indeterminate $u$ with integer coefficients. To each ordered pair $(y, w) \in W \times W$, Kazhdan and Lusztig associated some $P_{y,w} \in \mathbb{Z}[u]$ (now known as a *Kazhdan–Lusztig polynomial*), which satisfies the conditions: $P_{y,w} = 0$ if $y \nleqslant w$, $P_{w,w} = 1$, and $\deg P_{y,w} \leqslant (1/2)(\ell(w) - \ell(y) - 1)$ if $y < w$ (see [2]). For $y < w$ in $W$, let $\mu(w, y) = \mu(y, w)$ be the coefficient of $u^{(1/2)(\ell(w)-\ell(y)-1)}$ in $P_{y,w}$. We denote $y$—$w$ if $\mu(y, w) \neq 0$.

Checking the relation $y$—$w$ for $y, w \in W$ usually involves very complicated computation of Kazhdan–Lusztig polynomials. But it becomes easy in some special case:

$$\text{If } y < x \text{ in } W \text{ satisfy } \ell(y) = \ell(x) - 1, \text{ then } y\text{—}x \text{ with } \mu(x, y) = 1. \tag{1.1.1}$$