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Asymptotic properties of infinite directed unions of local quadratic transforms

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ABSTRACT

Let $(R, \mathfrak{m})$ be a regular local ring of dimension at least 2. For each valuation domain birationally dominating R, there is an associated sequence $\{R_n\}$ of local quadratic transforms of R. We consider the case where this sequence $\{R_n\}_{n\geq 0}$ is infinite and examine properties of the integrally closed local domain $S = \bigcup_{n\geq 0} R_n$ in the case where S is not a valuation domain. For this sequence, there is an associated boundary valuation ring $V = \bigcup_{n\geq 0} \bigcap_{i\geq n} V_i$, where V_i is the order valuation ring of R_i. There exists a unique minimal proper Noetherian overring T of S. T is the regular Noetherian UFD obtained by localizing outside the maximal ideal of S and $S = V \cap T$. In the present paper, we define functions w and e, where w is the asymptotic limit of the order valuations and e is the limit of the orders of transforms of principal ideals. We describe V explicitly in terms of w and e and prove that V is either rank 1 or rank 2. We define an invariant τ associated to S that is either a positive real number or $+\infty$. If τ is finite, then S is archimedean and T is not local. In this case, the function w defines the rank 1 valuation overring W of V and W dominates S. The rational dependence of τ over $w(T^\times)$ determines whether S is completely integrally closed and whether V has rank 1. We give examples where S is completely integrally closed. If τ is infinite, then S is non-archimedean and T is local. In this case, the function e defines

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the rank 1 valuation overring E of V. The valuation ring E is a DVR and E dominates T, and in certain cases we prove that E is the order valuation ring of T.

1. Introduction and summary

Let $(R, \mathfrak{m})$ be a regular local ring and let $S = \bigcup_{n \geq 0} R_n$ be an infinite directed union of local quadratic transforms as in the abstract. In [11], the authors consider ideal-theoretic properties of the integral domain S. The ring S is local and integrally closed. Abhyankar proves in [1, Lemma 12, p. 337] that if $\dim R = 2$, then S is a valuation domain. However, if $\dim R \geq 3$, then S is no longer a valuation domain in general. In the case where $\dim R \geq 3$, David Shannon examines properties of S in [21]. This motivates the following definition.

Definition 1.1. Let R be a regular local ring with $\dim R \geq 2$ and let $\{R_n\}_{n \geq 0}$ be an infinite sequence of regular local rings, where $R = R_0$ and R_{n+1} is a local quadratic transform of R_n for each $n \geq 0$. Then $S = \bigcup_{n \geq 0} R_n$ is a *Shannon extension* of R.

Let S be a Shannon extension of $R = R_0$ and let F denote the field of fractions of R. Associated to each of the regular local rings $(R_i, \mathfrak{m}_i)$, there is a rank 1 discrete valuation ring V_i defined by the *order function* ord_{R_i}, where for $x \in R_i$, $\operatorname{ord}_{R_i}(x) = \sup\{n \mid x \in \mathfrak{m}_i^n\}$. The family $\{V_i\}_{i=0}^{\infty}$ determines a unique set

$$V = \bigcup_{n \geq 0} \bigcap_{i \geq n} V_i = \{a \in F \mid \operatorname{ord}_{R_i}(a) \geq 0 \text{ for } i \gg 0\}.$$

The set V consists of the elements in F that are in all but finitely many of the V_i. In [11], the authors prove that V is a valuation domain that birationally dominates S, and call V the *boundary valuation ring* of the Shannon extension S.

In Section 2, we review the concept of the transform of an ideal, and in Sections 3 and 4, we discuss previous results on Shannon extensions. Theorem 3.2 describes an intersection decomposition $S = V \cap T$ of a Shannon extension S, where V is the boundary valuation of S and T is the intersection of all the DVR overrings of R that properly contain S. In Setting 3.3, we fix notation to use throughout the remainder of the paper. In Discussion 4.2, we describe conditions in order that S be a valuation domain.

In Section 5 we consider asymptotic behavior of the family $\{\operatorname{ord}_{R_n}\}_{n \geq 0}$ of order valuations of a Shannon extension $S = \bigcup_{n \geq 0} R_n$. For nonzero $a \in S$, we fix some n such that $a \in R_n$ and define $\mathrm{e}(a) = \lim_{i \to \infty} \operatorname{ord}_{R_{n+i}}((aR_n)^{R_{n+i}})$, where $(aR_n)^{R_{n+i}}$ denotes the transform in R_{n+i} of the ideal aR_n. For nonzero elements $a, b \in S$, we define $\mathrm{e}(\frac{a}{b}) = \mathrm{e}(a) - \mathrm{e}(b)$. In Lemmas 5.2 and 5.4, we prove the function e is well defined and that e describes factorization properties of elements in S.