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A non-Golod ring with a trivial product on its Koszul homology

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We present a monomial ideal $\mathfrak{a} \subset S$ such that $S/\mathfrak{a}$ is not Golod, even though the product in its Koszul homology is trivial. This constitutes a counterexample to a well-known result by Berglund and Jöllenbeck (the error can be traced to a mistake in an earlier article by Jöllenbeck).

On the positive side, we show that if R is a monomial ring such that the r-ary Massey product vanishes for all $r \leq \max(2, \operatorname{reg} R - 2)$, then R is Golod. In particular, if R is the Stanley–Reisner ring of a simplicial complex of dimension at most 3, then R is Golod if and only if the product in its Koszul homology is trivial.

Moreover, we show that if Δ is a triangulation of a $\Bbbk$-orientable manifold whose Stanley–Reisner ring is Golod, then Δ is 2-neighborly. This extends a recent result of Iriye and Kishimoto.

1. Introduction

Let $S = \Bbbk[x_1, \ldots, x_n]$ be a polynomial ring over some field $\Bbbk$, endowed with the fine $\mathbb{Z}^n$-grading, and let $\mathfrak{a} \subset S$ be a monomial ideal. The (multigraded) *Betti–Poincaré* series of $R := S/\mathfrak{a}$ is the formal power series

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$$P_{\Bbbk}^R(t, z_1, \ldots, z_n) := \sum_{j\geq 0} \sum_{\mathbf{a}\in\mathbb{N}^n} \dim_{\Bbbk} \operatorname{Tor}_j^R(\Bbbk, \Bbbk)_{\mathbf{a}} t^j z_1^{a_1} \cdots z_n^{a_n},$$

where $\operatorname{Tor}_j^R(\Bbbk, \Bbbk)_{\mathbf{a}}$ denotes the homogeneous component of $\operatorname{Tor}_j^R(\Bbbk, \Bbbk)$ in multidegree $\mathbf{a}$. We further consider the formal power series

$$Q_{\Bbbk}^R(t, z_1, \ldots, z_n) := \frac{\prod_{i=1}^n (1 + tz_i)}{1 - \sum_{j\geq 1} \sum_{\mathbf{a}\in\mathbb{N}^n} \dim_{\Bbbk} H_j(K_R)_{\mathbf{a}} t^{j+1} z_1^{a_1} \cdots z_n^{a_n}},$$

where K_R denotes the Koszul complex of R. The ring R is called a *Golod ring* if

$$P_{\Bbbk}^R(t, 1, \ldots, 1) = Q_{\Bbbk}^R(t, 1, \ldots, 1) \tag{1}$$

As reported by Golod in [18], Serre proved that every ring R satisfies the coefficient-wise inequality $P_{\Bbbk}^R(t, z_1, \ldots, z_n) \leq Q_{\Bbbk}^R(t, z_1, \ldots, z_n)$. Therefore, the Golod property is equivalent to the seemingly stronger condition that

$$P_{\Bbbk}^R(t, z_1, \ldots, z_n) = Q_{\Bbbk}^R(t, z_1, \ldots, z_n).$$

In the same article, Golod showed that R satisfies (1) if and only if the product and all higher Massey products on the Koszul homology $H_*(K_R)$ are trivial. Here, we say that the product is *trivial* if the product of every two elements of positive homological degrees is zero, and the higher Massey products are trivial if they are all defined and contain only zero.

The main contribution of the present article is an example of a monomial ideal $\mathfrak{a} \subset S$ such that the product in $H_*(K_{S/\mathfrak{a}})$ is trivial, but $S/\mathfrak{a}$ is not Golod. The example is given in Theorem 3.1. As far as we know, this is the first example of a non-Golod ring R with a trivial product in $H_*(K_R)$.

Our example provides a counterexample to a claim by Berglund and Jöllenbeck:

Claim 1.1 *(Theorem 5.1, [7]). Let $\mathfrak{a} \subset S$ be a monomial ideal and let $R := S/\mathfrak{a}$. Then the following are equivalent:*

(1) R is Golod.
(2) The product in the Koszul homology of R is trivial.

We would like to point out that the claim in [7] fails because its proof builds on an incorrect result of [24]. A list of special cases (both new and known) in which Claim 1.1 does hold is collected in Theorem 6.3.

The Golod property of quotients by monomial ideals is related to certain topological features of *moment-angle complexes*. Indeed, if $\mathfrak{a}$ is a squarefree monomial ideal, then it can be interpreted as the Stanley–Reisner ideal of a simplicial complex Δ. The moment-angle complex $\mathcal{Z}_\Delta$ is a certain topological space associated to Δ, which was introduced