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Quantum polynomial functors

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A B S T R A C T

We construct a category of quantum polynomial functors which deforms Friedlander and Suslin's category of strict polynomial functors. The main aim of this paper is to develop from first principles the basic structural properties of this category (duality, projective generators, braiding etc.) in analogy with classical strict polynomial functors. We then apply the work of Hashimoto and Hayashi in this context to construct quantum Schur/Weyl functors, and use this to provide new and easy derivations of quantum $(\mathrm{GL}_m, \mathrm{GL}_n)$ duality, along with other results in quantum invariant theory.

1. Introduction

Let $\Bbbk$ be a commutative ring and choose $q \in \Bbbk^\times$. The category $\mathcal{P}_q^d$ of quantum polynomial functors of homogeneous degree d consists of functors $\Gamma_q^d\mathcal{V} \to \mathcal{V}$, where $\mathcal{V}$ is the category of finite projective $\Bbbk$-modules, and $\Gamma_q^d\mathcal{V}$ is the category with objects natural numbers and morphisms given by

$$\mathrm{Hom}_{\Gamma_q^d\mathcal{V}}(m,n) := \mathrm{Hom}_{\mathcal{B}_d}(V_m^{\otimes d}, V_n^{\otimes d}).$$

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Here $\mathcal{B}_d$ is the Artin braid group, V_m denotes the free $\Bbbk$-module of rank n and the action of $\mathcal{B}_d$ on $V_m^{\otimes d}$ is given in Section 2.2. We think of the category $\Gamma_q^d\mathcal{V}$ as the category of standard Yang–Baxter spaces (V_n, R_n) (Section 2.2), and the morphisms can be viewed as degree d regular functions on the quantum Hom-space between standard Yang–Baxter spaces (although, as is usual in quantum algebra, only the regular functions are actually defined).

The purpose of this paper is to develop the basic structure theory of the category $\mathcal{P}_q^d$ in analogy with Friendlander and Suslin's work [8]. We first need to develop a theory of quantum linear algebra in great generality using Yang–Baxter spaces, and this is undertaken in Section 2.

We define morphisms between Yang–Baxter spaces over an algebra, and provide a universal characterization of quantum Hom-space algebra in Lemma 2.4. From this formalism we can derive many results about quantum Hom-space algebras functorialy. In particular, the dual of the Hom-space between two Yang–Baxter spaces of degree d is identified with certain braid group intertwiners, generalizing a well-known description of q-Schur algebras (Proposition 2.7). We construct the algebra of quantum $m \times n$ matrix space by specializing this theory to standard Yang–Baxter spaces.

We note that when the Yang–Baxter spaces are equal the quantum Hom space algebras have appeared in [19,14], but in the generality studied here these are new. We further remark that the general formalism of quantum linear algebra we develop builds on the work of Hashimoto–Hayashi [14], but it is not the same. They only consider the quantum Hom-space algebra between the same Yang–Baxter spaces, whereas for us it is crucial to build in morphisms between different Yang–Baxter spaces.

After the basic of quantum multilinear algebra are in place, we set out to develop the theory of quantum polynomial functors. To begin, the functor $\Gamma_q^{d,m} : \Gamma_q^d\mathcal{V} \to \mathcal{V}$ given by $n \mapsto \mathrm{Hom}_{\mathcal{B}_d}(V_m^{\otimes d}, V_n^{\otimes d})$ is called the quantum divided power functor. Theorem 4.7 states that $\Gamma_q^{d,m}$ is a projective generator of $\mathcal{P}_q^d$ when $m \geq d$. This uses a finite generation property for quantum polynomial functors, which we prove in Proposition 4.5.

Theorem 4.7 has several corollaries. It implies for instance that when $n \geq d$ we have an equivalence

$$\mathcal{P}_q^d \cong \mathsf{mod}(S_q(n, d)),$$

between the category of quantum polynomial functors of degree d and the category of modules over the q-Schur algebra $S_q(n, d)$ that are finite projective over $\Bbbk$. It also allows us to construct functors which represent weight spaces for representation of the quantum general linear group (Corollary 4.10). We note that from this and Proposition 2.7 one can immediately deduce the double centralizer property of Jimbo–Schur Weyl duality (Corollary 4.11).

In fact Theorem 4.7 is also needed to show that the R-matrix of the quantum general linear group are suitably functorial, that is they are natural with respect to morphisms in $\Gamma_q^d\mathcal{V}$, and thereby define a braiding on $\mathcal{P}_q := \bigoplus_{d=0}^{\infty} \mathcal{P}_q^d$ (Theorem 5.2). We emphasize