Journal of Algebra 479 (2017) 368–379

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Examples of Ore extensions which are maximal orders whose based rings are not maximal orders

Hidetoshi Marubayashi [a], Akira Ueda [b,*]

[a] *Department of Mathematics, Naruto University of Education, Naruto, Tokushima, 772-8502, Japan*
[b] *Department of Mathematics, Shimane University, Matsue, Shimane 690-8504, Japan*

A R T I C L E I N F O

Article history:
Received 8 August 2015
Available online 1 February 2017
Communicated by Louis Rowen

Keywords:
Ore extension
Maximal order
Hereditary Noetherian prime ring
Unique factorization ring

A B S T R A C T

Let R be a prime Goldie ring and (σ, δ) be a skew derivation on R. It is well known that if R is a maximal order, then the Ore extension $R[x; \sigma, \delta]$ is a maximal order. It was a long standing open question that the converse is true or not in case $\sigma \neq 1$ and $\delta \neq 0$.
We give an example of non-maximal order R with a skew derivation (σ, δ) on R ($\sigma \neq 1, \delta \neq 0$) such that $R[x; \sigma, \delta]$ is a maximal order.

1. Introduction

Let D be a hereditary Noetherian prime ring (an HNP ring for short) satisfying the following:

(a) there is a cycle $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ $(n \geq 2)$ such that $\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n = aD = Da$ for some $a \in D$ and
(b) any maximal ideal $\mathfrak{n}$ different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$ is invertible.

* Corresponding author.
E-mail addresses: marubaya@naruto-u.ac.jp (H. Marubayashi), ueda@riko.shimane-u.ac.jp (A. Ueda).

http://dx.doi.org/10.1016/j.jalgebra.2017.01.032
0021-8693/

We define a skew derivation (σ, δ) on D by $\sigma(r) = ara^{-1}$ and $\delta(r) = 0$ for all $r \in D$. Here by a *skew derivation* (σ, δ) on a ring S we mean σ is an automorphism of S and δ is a left σ-derivation on S.

Let $R = D[t]$ be the polynomial ring over D in an indeterminate t. Then (σ, δ) on D is extended to a skew derivation on R by $\sigma(t) = t$ and $\delta(t) = a$ (see [4]).

The aim of this paper is to obtain that the Ore extension $R[x; \sigma, \delta]$ is a maximal order and R is not a maximal order (Theorem 3.7).

For example, let $D = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, where $\mathbb{Z}$ is the ring of integers and p is a prime number. Then D is an HNP ring and $\mathfrak{m}_1 = \begin{pmatrix} p\mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $\mathfrak{m}_2 = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} \\ \mathbb{Z} & p\mathbb{Z} \end{pmatrix}$ is a cycle with $\mathfrak{m}_1 \cap \mathfrak{m}_2 = aD = Da$, where $a = \begin{pmatrix} p & p \\ 1 & 0 \end{pmatrix}$. On the other hand, $\left\{ \mathfrak{m}_q = \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} D \mid q \text{ is prime} \neq p \right\}$ is the full set of maximal ideals of D different from $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Then $R = D[t]$ is not a maximal order and the skew derivation (σ, δ) on R is as follows:

$$\sigma(f(t)) = \sigma(a_n)t^n + \cdots + \sigma(a_1)a^{-1}t + \sigma(a_0),$$

$$\delta(f(t)) = n\sigma(a_n)at^{n-1} + \cdots + \sigma(a_1)a,$$

where $f(t) = a_n t^n + \cdots + a_1 t + a_0 \in R$ (see Lemma 2.1) and the Ore extension $R[x; \sigma, \delta]$ is a maximal order.

Section 2 contains preliminary results which are used in Section 3.

In Section 3, we describe the structure of prime invertible ideals of $R[x; \sigma, \delta]$ (Proposition 3.2) and Theorem 3.7 is proved by showing that any v-ideal is v-invertible.

We refer the readers to [12] and [13] for terminology not defined in the paper.

2. Preliminary results

Let D be a ring with quotient ring $K = Q(D)$, σ an inner automorphism induced by a regular element a of D, that is, $\sigma(r) = ara^{-1}$ for all $r \in D$ and δ a trivial left σ-derivation on D, that is, $\delta(r) = 0$ for all $r \in D$.

Put $R = D[t]$, the polynomial ring over D in an indeterminate t. σ and δ are extended to an automorphism σ of R and a left σ-derivation δ on R as follows ([4, Lemma 1.2]);

$$\sigma(t) = t \quad \text{and} \quad \delta(t) = a.$$

It is well-known that a skew derivation (σ, δ) can be naturally extended to a skew derivation on K ([12, p. 132]).