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K_2 of Kac–Moody groups

Matthew Westaway

Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom

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ABSTRACT

Ulf Rehmann and Jun Morita, in their 1989 paper *A Matsumoto Type Theorem for Kac–Moody Groups*, gave a presentation of $K_2(A, F)$ for any generalised Cartan matrix A and field F. The purpose of this paper is to use this presentation to compute $K_2(A, F)$ more explicitly in the case when A is hyperbolic. In particular, we shall show that these $K_2(A, F)$ can always be expressed as a product of quotients of $K_2(F)$ and $K_2(2, F)$. Along the way, we shall also prove a similar result in the case when A has an odd entry in each column.

1. Introduction

In 1971, John Milnor introduced the functor K_2, which assigns to each field F an abelian group $K_2(F)$. The definition was straightforward: for $n \geq 3$, $K_2(n, F)$ was simply the kernel of the natural homomorphism from the Steinberg group $St(n, F)$ (cf. [3]) into $GL(n, F)$, and then $K_2(F)$ was defined as the direct limit of the $K_2(n, F)$. In fact, Milnor further proved that $K_2(n, F)$ was isomorphic to $K_2(F)$ for all $n \geq 3$.

While Milnor did not deal with the $n = 2$ case, a small change to the definition of the Steinberg group in this case yields an analogous group $K_2(2, F)$ (cf. [10]). So for each

E-mail address: m.p.westaway@warwick.ac.uk.

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field F we have two groups of interest: $K_2(F)$ and $K_2(2, F)$. In [6], Matsumoto was able to give presentations for both of these groups in terms of generators and relations, using ideas from root systems.

Once Tits introduced the idea of a Kac–Moody Group in [9], a generalisation of K_2 was born, one which was related to the root systems of arbitrary Kac–Moody algebras as opposed to just the finite ones which the original K_2 had been (cf. §6, §7 in [5], §1 in [4]). As a natural progression, Rehmann and Morita in [4] similarly generalised Matsumoto's seminal theorem in [6] to give a presentation of this new $K_2(A, F)$ for *all* Kac–Moody Lie algebras A and fields F.

This presentation is similar to Matsumoto's, but due to the greater generality of the root systems involved it is inherently more complicated. As a result, what $K_2(A, F)$ actually is for a given Generalised Cartan Matrix (GCM) A and field F is more difficult to determine than we would ideally like. Hence, in this paper we aim to give an alternate presentation for $K_2(A, F)$ for the hyperbolic GCMs, which should be easier to work with. In particular, it reduces to understanding the structure of the groups $K_2(F)$ and $K_2(2, F)$, a subject in which a fair amount of work has already been done.

This paper shall start by introducing the key definitions, theorems and notation which shall be used throughout. Next, in Section 3, we will explicate some of aspects of Rehmann and Morita's paper in more detail, in order to give us some more tools to work with, and we will use these to derive some straightforward results in Section 4. In Section 5, we shall give a general result which holds for all GCMs of a certain form which appears very frequently for hyperbolic GCMS. Then, we start computing $K_2(A, F)$ for the 2×2 hyperbolic GCMs (in Section 6), and then for the 3×3 matrices (in Section 7) and so on (concluding with Section 8), until all the hyperbolic cases are covered.

2. Preliminaries

An indecomposable *Generalised Cartan Matrix* (GCM) is called hyperbolic if it is not finite or affine, but any proper principal minor of the matrix is finite or affine (cf. [2]). In this paper, out goal is to compute $K_2(A, F)$ for any hyperbolic GCM A and field F.

We start by recalling the appropriate presentations given by Matsumoto [6] and Rehmann and Morita [4].

Theorem 2.1. *If A is a finite GCM (i.e. a Cartan matrix) and F is a field, then*

$$K_2(A, F) = \begin{cases} K_2(F) & \text{if } A \neq C_n (n \geq 1) \\ K_2(2, F) & \text{if } A = C_n (n \geq 1) \end{cases}$$

where C_n is used as in the standard notation for the finite-dimensional simple Lie algebras (with $C_1 = A_1$ and $C_2 = B_2$). Furthermore, these groups have the following presentations.

$K_2(F)$ is the abelian group generated by the symbols $\{u, v\}$ for $u, v \in F^*$ with defining relations: