Accepted Manuscript

Homotopy theory and generalized dimension subgroups

Sergei O. Ivanov, Roman Mikhailov, Jie Wu

PII: S0021-8693(17)30252-1
DOI: http://dx.doi.org/10.1016/j.jalgebra.2017.04.012
Reference: YJABR 16199

To appear in: *Journal of Algebra*

Received date: 27 May 2016

Please cite this article in press as: S.O. Ivanov et al., Homotopy theory and generalized dimension subgroups, *J. Algebra* (2017), http://dx.doi.org/10.1016/j.jalgebra.2017.04.012

HOMOTOPY THEORY AND GENERALIZED DIMENSION SUBGROUPS

SERGEI O. IVANOV, ROMAN MIKHAILOV, AND JIE WU

ABSTRACT. Let G be a group and R, S, T its normal subgroups. There is a natural extension of the concept of commutator subgroup for the case of three subgroups $\|R, S, T\|$ as well as the natural extension of the symmetric product $\|\mathbf{r}, \mathbf{s}, \mathbf{t}\|$ for corresponding ideals $\mathbf{r}, \mathbf{s}, \mathbf{t}$ in the integral group ring $\mathbb{Z}[G]$. In this paper, it is shown that the generalized dimension subgroup $G \cap (1 + \|\mathbf{r}, \mathbf{s}, \mathbf{t}\|)$ has exponent 2 modulo $\|R, S, T\|$. The proof essentially uses homotopy theory. The considered generalized dimension quotient of exponent 2 is identified with a subgroup of the kernel of the Hurewicz homomorphism for the loop space over a homotopy colimit of classifying spaces.

1. Introduction

Let G be a group and $\mathbb{Z}[G]$ its integral group ring. Every two-sided ideal $\mathfrak{a}$ in the integral group ring $\mathbb{Z}[G]$ of a group G determines a normal subgroup

$$D(G, \mathbf{a}) := G \cap (1 + \mathbf{a})$$

of G. Such subgroups are called *generalized dimension subgroups*. The identification of generalized dimension subgroups is a fundamental problem in the theory of group rings. In general, given an ideal $\mathbf{a}$, the identification of $D(G, \mathbf{a})$ is very difficult, for a survey on the problems in this area see [12], [16].

The idea that *the generalized dimension subgroups are related to the kernels of Hurewicz homomorphisms of certain spaces* was discussed in [16], [17], however, in the cited sources, all application of homotopical methods to the problems of group rings were related to very special cases. In this paper, we apply homotopy theory for a purely group-theoretical result of a more general type, namely to the description of the exponent of generalized dimension quotient constructed for a triple of normal subgroups in any group G.

Let G be a group and R, S its normal subgroups. Denote $\mathbf{r} = (R-1)\mathbb{Z}[G]$, $\mathbf{s} = (S-1)\mathbb{Z}[G]$. It is proved in [2] that

$$D(G, \mathbf{rs} + \mathbf{sr}) = [R, S]. \tag{1}$$

The following question arises naturally: how one can generalize the result (1) to the case of three or more normal subgroups of G. Our main result is the following.

Theorem 1. *Let G be a group and R, S, T its normal subgroups. Denote*

$$\mathbf{r} = (R-1)\mathbb{Z}[G], \ \mathbf{s} = (S-1)\mathbb{Z}[G], \ \mathbf{t} = (T-1)\mathbb{Z}[G]$$

The main result (Theorem 1) is supported by Russian Scientific Foundation, grant N 14-21-00035. The last author is also partially supported by the Singapore Ministry of Education research grant (AcRF Tier 1 WBS No. R-146-000-222-112) and a grant (No. 11329101) of NSFC of China.