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Covariant functors and asymptotic stability

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ABSTRACT

Let R be a commutative Noetherian ring, I, J ideals of R and M a finitely generated R-module. Let F be a covariant R-linear functor from the category of finitely generated R-modules to itself. We first show that if F is coherent, then the sets $\operatorname{Ass}_R F(M/I^nM)$, $\operatorname{Ass}_R F(I^{n-1}M/I^nM)$ and the values $\operatorname{depth}_J F(M/I^nM)$, $\operatorname{depth}_J F(I^{n-1}M/I^nM)$ become independent of n for large n. Next, we consider several examples in which F is a rather familiar functor, but is not coherent or not even finitely generated in general. In these cases, the sets $\operatorname{Ass}_R F(M/I^nM)$ still become independent of n for large n. We then show one negative result where F is not finitely generated. Finally, we give a positive result where F belongs to a special class of functors which are not finitely generated in general, an example of which is the zeroth local cohomology functor.

1. Introduction

In this paper, we will extend two results on asymptotic stability by M. Brodmann. Let us begin by fixing some terminology. A ring will mean a commutative ring with

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unity, unless specified otherwise. For a ring R, we let $\mathrm{Mod}(R)$ denote the category of R-modules and $\mathrm{mod}(R)$ the category of finitely generated R-modules. A functor will mean a covariant functor. For a nonempty set X and a sequence of elements $\{x_n\}_{n \geqslant k}$ of X, we say that asymptotic stability holds for the elements x_n, or that the elements x_n stabilize, if the sequence $\{x_n\}_{n \geqslant k}$ is eventually constant.

For the rest of this section, we will let R be a Noetherian ring unless specified otherwise, $L, M, N \in \mathrm{mod}(R)$ and I, J be ideals of R. The background of our project can be traced back to one of Ratliff's papers.

Question 1.1. [1, Introduction] Suppose that R is a domain and P is a prime ideal of R. If $P \in \mathrm{Ass}_R(R/I^k)$ for some $k \geqslant 1$, is $P \in \mathrm{Ass}_R(R/I^n)$ for all large n ?

Brodmann [2, (9)] gave a negative answer to the question, but at the same time, he proved a related, by now well-known result. Using the notation established so far, we will state his first result that we are interested in.

Theorem 1.2. *[2, page 16] The sets $\mathrm{Ass}_R(M/I^nM)$ and $\mathrm{Ass}_R(I^{n-1}M/I^nM)$ stabilize.*

The second result that we are interested in is as follows.

Theorem 1.3. *[3, Theorems 2(i) and 12(i)] The values $\mathrm{depth}_J(M/I^nM)$ and $\mathrm{depth}_J(I^{n-1}M/I^nM)$ stabilize.*

Most of this paper will be related to Theorem 1.2. There have been numerous generalizations of the theorem over the years. Here are a few of them.[2]

Theorem 1.4. *[5, Theorem 1] The sets $\mathrm{Ass}_R \mathrm{Tor}_i^R(N, R/I^n)$ and $\mathrm{Ass}_R \mathrm{Tor}_i^R(N, I^{n-1}/I^n)$ stabilize for any $i \geqslant 0$.*

Theorem 1.5. *[4, Proposition 3.4] Let $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ be a complex. Suppose that $L' \subseteq L$, $M' \subseteq M$ and $N' \subseteq N$ are submodules such that $\alpha(L') \subseteq M'$ and $\beta(M') \subseteq N'$. For $n \geqslant 0$, let $\mathrm{H}(n)$ denote the homology of the induced complex*

$$\frac{L}{I^nL'} \xrightarrow{\alpha_n} \frac{M}{I^nM'} \xrightarrow{\beta_n} \frac{N}{I^nN'}$$

Then the sets $\mathrm{Ass}_R \mathrm{H}(n)$ stabilize.

Corollary 1.6. *[4, Corollary 3.5] Let $M' \subseteq M$ be a submodule. Then for any $i \geqslant 0$, the sets $\mathrm{Ass}_R \mathrm{Tor}_i^R(N, M/I^nM')$ and $\mathrm{Ass}_R \mathrm{Ext}_R^i(N, M/I^nM')$ stabilize.*

[2] Although the theorems quoted here are related, the authors of [4] and [5] did not seem to know about the results of each other.