Journal of Algebra 484 (2017) 265–274

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

A generalization of dual symmetry and reciprocity for symmetric algebras

Taro Sakurai

Department of Mathematics and Informatics, Graduate School of Science, Chiba University, 1-33, Yayoi-cho, Inage-ku, Chiba-shi, Chiba, 263-8522, Japan

A R T I C L E   I N F O

Article history:
Received 17 May 2016
Available online 2 May 2017
Communicated by Michel Broué

MSC:
16P10
16D40
18G05
20C05
20C20

Keywords:
Loewy structure
Radical layer
Socle layer
Symmetric algebra
Socle functor
Capital functor

A B S T R A C T

Slicing a module into semisimple ones is useful to study modules. Loewy structures provide a means of doing so. To establish the Loewy structures of projective modules over a finite dimensional symmetric algebra over a field F, the Landrock lemma is a primary tool. The lemma and its corollary relate radical layers of projective indecomposable modules to radical layers of the F-duals of those modules ("dual symmetry") and to socle layers of those modules ("reciprocity").

We generalize these results to an arbitrary finite dimensional algebra A. Our main theorem, which is the same as the Landrock lemma for finite dimensional symmetric algebras, relates radical layers of projective indecomposable modules P to radical layers of the A-duals of those modules and to socle layers of injective indecomposable modules νP where ν is the Nakayama functor. A key tool to prove the main theorem is a pair of adjoint functors, which we call socle functors and capital functors.

E-mail address: tsakurai@math.s.chiba-u.ac.jp.

http://dx.doi.org/10.1016/j.jalgebra.2017.04.013

1. Introduction

Semisimple modules are one of the most well-understood classes of modules. Hence slicing a module into semisimple ones is a natural way to study modules. Loewy structures provide a means of doing so. To establish Loewy structures several studies have been done [1,2,4,11]. A primary tool is the Landrock lemma [5,6], which is stated below.

Let A be a finite dimensional algebra over a field F and $(-)^* \coloneqq \mathrm{Hom}_F(-, F)$ the *F-dual functor*. The opposite algebra is denoted by A^{op}. The term module refers to a finitely generated right module. Recall that A is a *symmetric algebra* if $A \cong A^*$ as (A, A)-bimodules. For other notations see Definition 2.1.

Theorem 1.1. (See Landrock [5, Theorem B] for (i).) *For a finite dimensional symmetric algebra A over a field F, let P_i and P_j be the projective covers of simple A-modules S_i and S_j respectively.*

(i) *For an integer $n \geq 1$ we have an F-linear isomorphism*

$$\mathrm{Hom}_A(\mathrm{rad}_n\, P_i, S_j) \cong \mathrm{Hom}_{A^{\mathrm{op}}}\big(\mathrm{rad}_n(P_j^*), S_i^*\big).$$

(ii) *For an integer $n \geq 1$ we have an F-linear isomorphism*

$$\mathrm{Hom}_A(\mathrm{rad}_n\, P_i, S_j) \cong \mathrm{Hom}_A(S_i, \mathrm{soc}_n\, P_j).$$

This theorem have been incredibly useful in the study of Loewy structures of symmetric algebras (such as finite group algebras, see [2,4,11]). The purpose of this paper is to generalize this useful theorem to the study of arbitrary finite dimensional algebras. To state our main theorem we let $(-)^\vee \coloneqq \mathrm{Hom}_A(-, A)$ be the *A-dual functor* and $\nu(-) \coloneqq \big((-)^\vee\big)^*$ the *Nakayama functor*.

Theorem 1.2. *For a finite dimensional algebra A over a field F, let P_i and P_j be the projective covers of simple A-modules S_i and S_j respectively.*

(i) *For an integer $n \geq 1$ we have an F-linear isomorphism*

$$\mathrm{Hom}_A(\mathrm{rad}_n\, P_i, S_j) \cong \mathrm{Hom}_{A^{\mathrm{op}}}\big(\mathrm{rad}_n(P_j^\vee), S_i^*\big).$$

(ii) *For an integer $n \geq 1$ we have an F-linear isomorphism*

$$\mathrm{Hom}_A(\mathrm{rad}_n\, P_i, S_j) \cong \mathrm{Hom}_A(S_i, \mathrm{soc}_n\, \nu P_j).$$

This paper is organized as follows. In Section 2 we introduce key tools to prove our main theorem, socle functors and capital functors. We then prove some useful lemmas. Section 3 is devoted to prove our main theorem. We also derive Theorem 1.1 from our