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Real rank two geometry

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The real rank two locus of an algebraic variety is the closure of the union of all secant lines spanned by real points. We seek a semi-algebraic description of this set. Its algebraic boundary consists of the tangential variety and the edge variety. Our study of Segre and Veronese varieties yields a characterization of tensors of real rank two.

1. Introduction

Low-rank approximation of tensors is a fundamental problem in applied mathematics [3,5]. We here approach this problem from the perspective of real algebraic geometry. Our goal is to give an exact semi-algebraic description of the set of tensors of real rank two and to characterize its boundary. This complements the results on tensors of non-negative rank two presented in [1], and it offers a generalization to the setting of arbitrary varieties, following [2].

A familiar example is that of $2 \times 2 \times 2$-tensors (x_{ijk}) with real entries. Such a tensor lies in the closure of the real rank two tensors if and only if the *hyperdeterminant* is non-negative:

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$$
\begin{aligned}
&x_{000}^2x_{111}^2 + x_{001}^2x_{110}^2 + x_{010}^2x_{101}^2 + x_{011}^2x_{100}^2 + 4x_{000}x_{011}x_{101}x_{110} + 4x_{001}x_{010}x_{100}x_{111} \\
&\quad - 2x_{000}x_{001}x_{110}x_{111} - 2x_{000}x_{010}x_{101}x_{111} - 2x_{000}x_{011}x_{100}x_{111} \\
&\quad - 2x_{001}x_{010}x_{101}x_{110} - 2x_{001}x_{011}x_{100}x_{110} - 2x_{010}x_{011}x_{100}x_{101} \;\geq\; 0.
\end{aligned}
\tag{1}
$$

If this inequality does not hold then the tensor has rank two over $\mathbb{C}$ but rank three over $\mathbb{R}$.

To understand this example geometrically, consider the *Segre variety* $X = \mathrm{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$, i.e. the set of rank one tensors, regarded as points in the projective space $\mathbb{P}^7 = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. The hyperdeterminant defines a quartic hypersurface $\tau(X)$ in $\mathbb{P}^7$. The real projective space $\mathbb{P}^7_{\mathbb{R}}$ is divided into two connected components by its real points $\tau(X)_{\mathbb{R}}$. One of the two connected components is the locus $\rho(X)$ that comprises the real rank two tensors.

This paper views real rank in a general geometric framework, studied recently by Blekherman and Sinn [2]. Let X be an irreducible variety in a complex projective space $\mathbb{P}^N$ that is defined over $\mathbb{R}$ and whose set $X_{\mathbb{R}} = X \cap \mathbb{P}^N_{\mathbb{R}}$ of real points is Zariski dense in X. The *secant variety* $\sigma(X)$ is the closure of the set of points in $\mathbb{P}^N$ that lie on a line spanned by two points in X. The *tangential variety* $\tau(X)$ is a subvariety of the secant variety. Namely, $\tau(X)$ is the closure of the set of points in $\mathbb{P}^N$ that lie on a tangent line to X at a smooth point. In this paper closure is taken with respect to the Euclidean topology, unless otherwise specified. For the secant and tangential varieties above, the Euclidean closure and Zariski closure coincide.

Our object of interest is the *real rank two locus* $\rho(X)$. This is a semi-algebraic set in the real projective space $\mathbb{P}^N_{\mathbb{R}}$. We define $\rho(X)$ as the (Euclidean) closure of the set of points that lie on a line spanned by two points in $X_{\mathbb{R}}$. Our hypotheses ensure that $\rho(X)$ is Zariski dense in $\sigma(X)$. The inclusion of the closed set $\rho(X)$ in the real secant variety $\sigma(X)_{\mathbb{R}}$ is usually strict. The difference consists of points of X-rank two whose real X-rank exceeds two.

Two varieties most relevant for applications are the *Segre variety* $X = \mathrm{Seg}(\mathbb{P}^{n_1-1} \times \cdots \times \mathbb{P}^{n_d-1})$ and the *Veronese variety* $X = \nu_d(\mathbb{P}^{n-1})$. The ambient dimensions are $N = n_1 \cdots n_d - 1$ and $N = \binom{n+d-1}{d} - 1$, and X consists of (symmetric) tensors of rank one. The secant variety $\sigma(X)$ is the closure of the set of tensors of complex rank two, and $\sigma(X)_{\mathbb{R}}$ is the set of real points of that complex projective variety. The real rank two locus $\rho(X)$ is the closure of the tensors of real rank two. This is a subset of $\sigma(X)_{\mathbb{R}}$. The containment is strict when $d \geq 3$.

It is instructive to examine the case of $3 \times 2 \times 2$-tensors. The secant variety $\sigma(X)$ has dimension 9 in $\mathbb{P}^{11}$. By [13], it consists of all tensors whose 3×4 matrix flattening satisfies

$$
\mathrm{rank} \begin{pmatrix} x_{000} & x_{001} & x_{010} & x_{011} \\ x_{100} & x_{101} & x_{110} & x_{111} \\ x_{200} & x_{201} & x_{210} & x_{211} \end{pmatrix} \;\leq\; 2. \tag{2}
$$