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Discrete polymatroids satisfying a stronger symmetric exchange property

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ABSTRACT

In this paper we introduce discrete polymatroids satisfying the one-sided strong exchange property and show that they are sortable (as a consequence their base rings are Koszul) and that they satisfy White's conjecture. Since any pruned lattice path polymatroid satisfies the one-sided strong exchange property, this result provides an alternative proof for one of the main theorems of J. Schweig in [12], where it is shown that every pruned lattice path polymatroid satisfies White's conjecture. In addition we characterize a class of such polymatroids whose base rings are Gorenstein. Finally for two classes of pruned lattice path polymatroidal ideals I and their powers we determine their depth and their associated prime ideals, and furthermore determine the least power k for which depth S/I^k and $\mathrm{Ass}(S/I^k)$ stabilize. It turns out that depth S/I^k stabilizes precisely when $\mathrm{Ass}(S/I^k)$ stabilizes in both cases.

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Introduction

Throughout this paper, we always denote vectors in boldface such as $\mathbf{u}, \mathbf{v}, \mathbf{u}_i, \mathbf{v}_i, \boldsymbol{\alpha}$ and etc. If $\mathbf{u}$ is a vector in $\mathbb{Z}^n$, we use either u_i or $\mathbf{u}(i)$ to denote its ith entry and use $\mathbf{u}(A)$ to denote the number $\sum_{i \in A} u_i$ for a subset $A \subseteq \{1, \ldots, n\}$. Let k, ℓ be integers with $k \leq \ell$. Then $[k, \ell]$ denotes the interval $\{k, k+1, \ldots, \ell\}$ and $[1, k]$ is usually denoted by $[k]$ for short. Also we denote by $\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_n$ the canonical basis of $\mathbb{Z}^n$ and by $\mathbb{Z}_+$ the set of non-negative integers. The set $\mathbb{Z}_+^n$ has a partial ordering "$\leq$" defined by:

$$\mathbf{u} \leq \mathbf{v} \iff u_i \leq v_i \text{ for each } i = 1, \ldots, n.$$

Unless otherwise stated, S always stands for the polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ over a field $\mathbb{K}$. For a subset $A \subseteq [n]$, P_A denotes the monomial prime ideal $(x_i\colon\ i \in A)$ of S.

For the basic knowledge of matroids we refer to [11]. In [5], discrete polymatroids are introduced, which generalize matroids in the way that monomial ideals generalize squarefree monomial ideals.

A *discrete polymatroid* on the ground set $[n]$ is a nonempty finite set $\mathbf{P} \subseteq \mathbb{Z}_+^n$ satisfying

(D1) if $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{P}$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{Z}_+^n$ with $\mathbf{v} \leq \mathbf{u}$, then $\mathbf{v} \in \mathbf{P}$;
(D2) if $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbf{P}$ and $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{P}$ with $|\mathbf{u}| < |\mathbf{v}|$, then there is $i \in [n]$ with $u_i < v_i$ such that $\mathbf{u} + \boldsymbol{\varepsilon}_i \in \mathbf{P}$. Here $|\mathbf{u}| := \mathbf{u}([n])$.

A *base* of a discrete polymatroid $\mathbf{P}$ is a vector $\mathbf{u}$ of $\mathbf{P}$ such that $\mathbf{u} < \mathbf{v}$ for no $\mathbf{v} \in \mathbf{P}$. Every base of $\mathbf{P}$ has the same modulus rank($\mathbf{P}$), the *rank* of $\mathbf{P}$. Let $B(\mathbf{P})$ or simply B denote the set of bases of $\mathbf{P}$. Every discrete polymatroid satisfies the following symmetric exchange property: if $\mathbf{u}$ and $\mathbf{v}$ are vectors of B, then for any $i \in [n]$ with $u_i < v_i$, there is $j \in [n]$ with $u_j > v_j$ such that both $\mathbf{u} + \boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j$ and $\mathbf{v} - \boldsymbol{\varepsilon}_i + \boldsymbol{\varepsilon}_j$ belong to B. Conversely if B is a set of vectors in $\mathbb{Z}_+^n$ of the same modulus satisfying the symmetric exchange property, then $\mathbf{P} = \{\mathbf{u} \in \mathbb{Z}_+^n\colon\ \mathbf{u} \leq \mathbf{v} \text{ for some } \mathbf{v} \in B\}$ is a discrete polymatroid with B as its set of bases.

We are interested in two algebraic structures associated with a discrete polymatroid $\mathbf{P}$: its *base ring* and its *polymatroidal ideal*. Let $\mathbb{K}$ be a field. The base ring $\mathbb{K}[B(\mathbf{P})]$ (or simply $\mathbb{K}[B]$) of $\mathbf{P}$ is defined to be the subring of $\mathbb{K}[t_1, \ldots, t_n]$ generated by monomials $\mathbf{t}^{\mathbf{u}} = t_1^{u_1} \ldots t_n^{u_n}$ with $\mathbf{u} \in B$. Meanwhile the polymatroidal ideal of $\mathbf{P}$ is defined to be the monomial ideal in $S = \mathbb{K}[x_1, \ldots, x_n]$ generated by $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \ldots x_n^{u_n}$ with $\mathbf{u} \in B$.

Let T be the polynomial ring $\mathbb{K}[x_{\mathbf{u}} : \mathbf{u} \in B]$ and let I_B be the kernel of the $\mathbb{K}$-algebra homomorphism $\phi : T \to \mathbb{K}[B]$ with $\phi(x_{\mathbf{u}}) = \mathbf{t}^{\mathbf{u}}$ for any $\mathbf{u} \in B$. There are some obvious generators in I_B. Indeed, let $\mathbf{u}, \mathbf{v} \in B$ with $u_i > v_i$. Then there exists j such that $u_j < v_j$ and such that $\mathbf{u} - \boldsymbol{\varepsilon}_i + \boldsymbol{\varepsilon}_j$ and $\mathbf{v} + \boldsymbol{\varepsilon}_i - \boldsymbol{\varepsilon}_j$ belong to B. We see that $x_{\mathbf{u}} x_{\mathbf{v}} - x_{\mathbf{u}-\boldsymbol{\varepsilon}_i+\boldsymbol{\varepsilon}_j} x_{\mathbf{v}+\boldsymbol{\varepsilon}_i-\boldsymbol{\varepsilon}_j} \in I_B$. Such relations are called *symmetric exchange relations*. White [15] conjectured that for a matroid the symmetric exchange relations generate I_B. In [5], Herzog and Hibi predicted that this also holds for discrete polymatroids.