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Smooth varieties with torus actions ☆

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ABSTRACT

In this paper we provide a characterization of smooth algebraic varieties endowed with a faithful algebraic torus action in terms of a combinatorial description given by Altmann and Hausen. Our main result is that such a variety X is smooth if and only if it is locally isomorphic in the étale topology to the affine space endowed with a linear torus action. Furthermore, this is the case if and only if the combinatorial data describing X is locally isomorphic in the étale topology to the combinatorial data describing affine space endowed with a linear torus action. Finally, we provide an effective method to check the smoothness of a $\mathbb{G}_m$-threefold in terms of the combinatorial data.

Introduction

Let k be an algebraically closed field of characteristic zero and let $\mathbb{T}$ be the algebraic torus $\mathbb{T} = (\mathbb{G}_m)^k$ of dimension k where $\mathbb{G}_m$ is the multiplicative group of the field $(\mathrm{k}^*, \cdot)$. The variety $\mathbb{T}$ has a natural structure of algebraic group. We denote by M its character lattice and by N its 1-parameter subgroup lattice. In this paper a variety denotes an

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integral separated scheme of finite type. A $\mathbb{T}$-variety is a normal variety X endowed with a faithful action of $\mathbb{T}$ acting on X by regular automorphisms. The assumption that the $\mathbb{T}$-action on X is faithful is not a restriction since given any regular $\mathbb{T}$-action α, the kernel $\ker \alpha$ is a normal algebraic subgroup of $\mathbb{T}$ and $\mathbb{T}/(\ker \alpha)$ is again an algebraic torus acting faithfully on X.

The complexity of a $\mathbb{T}$-variety is the codimension of the generic orbit. Furthermore, since the action is assumed to be faithful, the complexity of X is given by $\dim X - \dim \mathbb{T}$. The best known example of $\mathbb{T}$-varieties are those of complexity zero, i.e., toric varieties. Toric varieties were first introduced by Demazure in [7] as a tool to study subgroups of the Cremona group. Toric varieties allow a combinatorial description in term of certain collections of strongly convex polyhedral cones in the vector space $N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ called fans, see for instance [19,12,6].

For higher complexity there is also a combinatorial description of a $\mathbb{T}$-variety. We will use the language of p-divisors first introduced by Altmann and Hausen in [2] for the affine case and generalized in [3] to arbitrary $\mathbb{T}$-varieties. This description, that we call the A-H combinatorial description, generalizes previously known partial cases such as [14,8,11,24]. See [4] for a detailed survey on the topic.

Since the introduction of the A-H description, a lot of work has been done in generalizing the known results from toric geometry to the more general case of $\mathbb{T}$-varieties. On of the most basic parts of the theory that are still open is the characterization of smooth $\mathbb{T}$-varieties of complexity higher than one. In particular, several classification of singularities of $\mathbb{T}$-varieties are given in [17], but no smoothness criterion is given in complexity higher than one. In this paper we achieve such a characterization in arbitrary complexity.

By Sumihiro's Theorem [23], every (normal) $\mathbb{T}$-variety admits an affine cover by $\mathbb{T}$-invariant affine open sets. Hence, to study smoothness, it is enough to consider affine $\mathbb{T}$-varieties. The A-H description of an affine $\mathbb{T}$-variety X consists in a couple $(Y, \mathcal{D})$, where Y is a normal semiprojective variety that is a kind of quotient of X, usually called the Chow quotient and $\mathcal{D}$ is a divisor on Y whose coefficients are not integers as usual, but polyhedra in $N_{\mathbb{Q}}$, see Section 1 for details.

It is well known that an affine toric variety X of dimension n is smooth if and only if it is equivariantly isomorphic to a $\mathbb{T}$-invariant open set in $\mathbb{A}^n$ endowed with the standard $\mathbb{T}$-action of complexity 0 by component-wise multiplication. Our main result states that an affine $\mathbb{T}$-variety X of dimension n and of arbitrary complexity is smooth if and only if it is locally isomorphic in the étale topology to the affine space $\mathbb{A}^n$ endowed with a linear $\mathbb{T}$-action, see Proposition 4. Furthermore, X is smooth if and only if the combinatorial data $(Y, \mathcal{D})$ is locally isomorphic in the étale topology to the combinatorial data $(Y', \mathcal{D}')$ of the affine space $\mathbb{A}^n$ endowed with a linear $\mathbb{T}$-action, see Theorem 7. The main ingredient in our result is Luna's Slice Theorem [18].

In order to effectively apply Theorem 7, it is necessary to know the A-H description of $\mathbb{A}^n$ endowed with all possible linear $\mathbb{T}$-actions. The case of complexity 0 and 1 are well known, see Corollary 9. In Proposition 12, we compute the combinatorial description of