Journal of Algebra 490 (2017) 256–282

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The Koszul property for spaces of quadrics of codimension three

Alessio D'Alì *

Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso 35, 16146 Genova, Italy

ARTICLE INFO

Article history:
Received 5 June 2016
Available online 25 July 2017
Communicated by Luchezar L. Avramov

MSC:
16S37
13P10
13F20

Keywords:
Koszul algebra
Quadratic Gröbner basis
Quadratic ideal
Space of quadrics
Hilbert series

ABSTRACT

In this paper we prove that, if $\Bbbk$ is an algebraically closed field of characteristic different from 2, almost all quadratic standard graded $\Bbbk$-algebras R such that $\dim_{\Bbbk} R_2 = 3$ are Koszul. More precisely, up to graded $\Bbbk$-algebra homomorphisms and trivial fiber extensions, we find out that only two (or three, when the characteristic of $\Bbbk$ is 3) algebras of this kind are non-Koszul.
Moreover, we show that there exist nontrivial quadratic standard graded $\Bbbk$-algebras with $\dim_{\Bbbk} R_1 = 4$, $\dim_{\Bbbk} R_2 = 3$ that are Koszul but do not admit a Gröbner basis of quadrics even after a change of coordinates, thus settling in the negative a question asked by Conca.

1. Introduction

A (commutative) standard graded algebra R over a field $\Bbbk$ is a quotient of a polynomial ring over $\Bbbk$ in a finite number of variables by a homogeneous ideal I not containing any linear form. We say that R is *Koszul* when the minimal graded free resolution of $\Bbbk$ as an

* Present address: Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstraße 22, 04103 Leipzig, Germany.
E-mail address: dali@dima.unige.it.

http://dx.doi.org/10.1016/j.jalgebra.2017.06.032

R-module is linear: for a survey about Koszulness in the commutative setting, see [7]. It is well-known that any Koszul algebra is quadratic, i.e. its defining ideal I is generated by quadrics. Note that, if R is a trivial fiber extension, i.e. it contains a linear form $\ell \in R_1$ such that $\ell R_1 = 0$, then R is Koszul if and only if R/ℓ is Koszul.

For the rest of this introduction, let R be a quadratic standard graded $\Bbbk$-algebra, where $\Bbbk$ is an algebraically closed field of characteristic different from two. Backelin proved in [1] that, if $\dim_{\Bbbk} R_2 = 2$, then R is Koszul (this actually holds for any field $\Bbbk$). Conca then proved in [5] that, if $\dim_{\Bbbk} R_2 = 3$ and R is Artinian, then R is Koszul. The main problem we are addressing in this paper is to find out what happens when, in the latter case, we drop the Artinian assumption. We will prove in Theorem 3.1 that, up to trivial fiber extension, the only non-Koszul algebras live in embedding dimension three. More precisely, these non-Koszul algebras in three variables are the two (or three, when $\operatorname{char} \Bbbk = 3$) objects exhibited by Backelin and Fröberg in [2]. As a byproduct, we shall also obtain a list of all possible Hilbert series when $\dim_{\Bbbk} R_2 = 3$, see Theorem 3.3.

Our results are in agreement with the numerical data presented by Roos in the characteristic 0 four-variable case, see [15].

Another interesting property one can investigate is whether or not R is G-quadratic, i.e. it admits a Gröbner basis of quadrics (possibly after a change of coordinates). It is well-known that G-quadraticness is a sufficient but not necessary condition for Koszulness, see [10, Section 6]. We will show in Corollary 5.3 that, in our context, there exist four-variable Koszul algebras which are not G-quadratic and are not trivial fiber extensions: this answers a question asked by Conca in [5, Section 4].

Computations made using the computer algebra system CoCoA [4] helped us to produce conjectures and gave us hints about the behaviour of the objects studied.

2. Tools and techniques

2.1. Koszulness and related concepts

We are going to recall very briefly some definitions to put our results into context. We direct the interested reader to the survey [7] for further information.

Let $\Bbbk$ be a field, S the polynomial ring $\Bbbk[x_1, \dots, x_n]$ and I a homogeneous ideal of S not containing any linear form. Let R be the standard graded $\Bbbk$-algebra S/I.

Definition 2.1. The (commutative) $\Bbbk$-algebra R is *Koszul* if the minimal graded free resolution of the residue field $\Bbbk$ as an R-module is linear.

It is usually very difficult to establish the Koszulness of a certain algebra by making use of the definition only. Because of this we will collect below some sufficient conditions, see Theorem 2.6.

Definition 2.2. If there exists a family $\mathfrak{F}$ of ideals of R such that:

- $(0) \in \mathfrak{F}$, $(R_1) \in \mathfrak{F}$,