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ON THE UNIFORM SPREAD OF ALMOST SIMPLE SYMPLECTIC AND ORTHOGONAL GROUPS

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Abstract. A group is $\frac{3}{2}$-generated if every non-identity element is contained in a generating pair. A conjecture of Breuer, Guralnick and Kantor from 2008 asserts that a finite group is $\frac{3}{2}$-generated if and only if every proper quotient of the group is cyclic, and recent work of Guralnick reduces this conjecture to almost simple groups. In this paper, we prove a stronger form of the conjecture for almost simple symplectic and odd-dimensional orthogonal groups. More generally, we study the uniform spread of these groups, obtaining lower bounds and related asymptotics. This builds on earlier work of Burness and Guest, who established the conjecture for almost simple linear groups.

1. Introduction

Let G be a finite group. We say that G is *d-generated* if G has a generating set of size d. It is well-known that every finite simple group is 2-generated [39, 2]. In fact, almost surely, any two elements of a finite simple group G generate the group, in the sense that the probability that two randomly chosen elements form a generating pair tends to one as $|G|$ tends to infinity [31, 35]. Therefore, generating pairs are abundant in finite simple groups, and it is natural to ask how they are distributed across the group. With this in mind, we say that G is $\frac{3}{2}$-*generated* if every non-identity element of G is contained in a generating pair. By a theorem of Guralnick and Kantor [27] (also see Stein [38]), every finite simple group is $\frac{3}{2}$-generated, resolving a question of Steinberg [39] in the affirmative.

It is straightforward to see that every proper quotient of a $\frac{3}{2}$-generated group is cyclic. In [9], Breuer, Guralnick and Kantor make the following remarkable conjecture.

Conjecture. *A finite group is $\frac{3}{2}$-generated if and only if every proper quotient is cyclic.*

This conjecture has recently been reduced by Guralnick [26] to almost simple groups G. By the main theorem of [21], these groups are 3-generated and, in fact, 2-generated if $G/\mathrm{soc}(G)$ is cyclic, where $\mathrm{soc}(G)$ denotes the (simple) socle of G. In the case where $\mathrm{soc}(G)$ is alternating the conjecture was established in [6], and the sporadic groups are handled in [9] using computational methods. Therefore, it remains to consider the almost simple groups of Lie type. In [17], Burness and Guest establish a stronger version of the conjecture for almost simple linear groups. The aim of this paper is to extend this result to almost simple symplectic and odd-dimensional orthogonal groups. We will handle the remaining groups of Lie type in a forthcoming paper.

Following Brenner and Wiegold [7], a finite group G has *spread* k if for any k non-identity elements $x_1, \dots, x_k \in G$ there exists $g \in G$ such that, for all i, $\langle x_i, g\rangle = G$. Moreover, G is said to have *uniform spread* k if the element g can be chosen from a fixed conjugacy class of G. We write $s(G)$ (respectively $u(G)$) for the greatest k such that G has spread k (respectively uniform spread k). (If G is cyclic, then write $s(G) = u(G) = \infty$.)

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