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Relative derived dimensions for cotilting modules

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For a Noetherian ring R and a cotilting R-module T of injective dimension at least 1, we prove that the derived dimension of R with respect to the category $\mathcal{X}_T$ is precisely the injective dimension of T by applying Auslander–Buchweitz theory and Ghost Lemma. In particular, when R is a commutative Noetherian Cohen–Macaulay local ring with a canonical module ω_R and $\dim R \geq 1$, the derived dimension of R with respect to the category of maximal Cohen–Macaulay modules is precisely $\dim R$.

1. Introduction

This paper is a companion to [1]. We give an explicit value of the relative dimension of the derived category with respect to the subcategory associated with a cotilting module.

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In this paper, we denote by R a Noetherian ring. All R-modules are finitely generated right R-modules. We denote by $\operatorname{mod} R$ the abelian category of R-modules and by $\mathsf{D}^{\mathrm{b}}(\operatorname{mod} R)$ the derived category of $\operatorname{mod} R$.

Then our main result is the following, which completes a main result Theorem 5.3 in [1].

Theorem 1.1. *Let R be a Noetherian ring and T a cotilting R-module (see Definition 2.1) with* $\operatorname{inj.dim} T \geq 1$*. Then we have an equality*

$$\mathcal{X}_T\text{–tri.dim } \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) = \operatorname{inj.dim} T.$$

The inequality $\leq$ was shown in [1, Theorem 5.3]. In this paper, we will prove the converse inequality by applying Auslander–Buchweitz theory and Ghost Lemma.

We apply Theorem 1.1 to the following settings. For a commutative Noetherian Cohen–Macaulay local ring R with a canonical module ω_R, we denote by $\mathsf{CM}R$ the category of maximal Cohen–Macaulay modules. We call an R-algebra Λ an *R-order* if $\Lambda \in \mathsf{CM}R$. We denote by $\mathsf{CM}\Lambda$ the category of maximal Cohen–Macaulay Λ-modules (i.e. Λ-modules X satisfying $X \in \mathsf{CM}R$). As a special case of Theorem 1.1, we obtain the following results, which completes the inequalities (1.2.1) and (4.2.1) in [1].

Corollary 1.2. *Let R be a commutative Noetherian Cohen–Macaulay local ring with a canonical module ω_R and* $\dim R \geq 1$*. Then*

(1) *We have an equality*

$$(\mathsf{CM}R)\text{–tri.dim } \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) = \dim R.$$

(2) *More generally, for an R-order Λ, we have an equality*

$$(\mathsf{CM}\Lambda)\text{–tri.dim } \mathsf{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) = \dim R.$$

Proof. Since ω_R (respectively, $\omega_\Lambda := \operatorname{Hom}_R(\Lambda, \omega_R)$) is a cotilting module with injective dimension $\dim R$, the assertion follows from Theorem 1.1. □

2. Preliminaries

In this section, we will introduce the concept of a cotilting module.

For an R-module T, we define the full subcategory ${}^{\perp}T$ of $\operatorname{mod} R$ as follows:

$${}^{\perp}T := \{X \in \operatorname{mod} R \mid \operatorname{Ext}^i_R(X, T) = 0 \text{ for any } i > 0\}.$$

Definition 2.1. An R-module T is called *cotilting* if it satisfies the following three conditions: