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Global Okounkov bodies for Bott–Samelson varieties ☆

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A B S T R A C T

We use the theory of Mori dream spaces to prove that the global Okounkov body of a Bott–Samelson variety with respect to a natural flag of subvarieties is rational polyhedral. As a corollary, Okounkov bodies of effective line bundles over Schubert varieties are shown to be rational polyhedral. In particular, it follows that the global Okounkov body of a flag variety G/B is rational polyhedral. As an application we show that the asymptotic behaviour of dimensions of weight spaces in section spaces of line bundles is given by the volume of polytopes.

Introduction

Okounkov bodies were first introduced by A. Okounkov in his famous paper [20] as a tool for studying multiplicities of group representations. The idea is that one should

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be able to approximate these multiplicities by counting the number of integral points in a certain convex body in $\mathbb{R}^n$. More precisely, the setting is the following. Let G be a complex reductive group which acts as automorphisms on an effective line bundle L over a projective variety X, and hence defines a representation on the space of sections $H^0(X, L^k)$ for each integral power, L^k, of L. For an n-dimensional variety, Okounkov constructs a convex compact set $\Delta \subseteq \mathbb{R}^n$ whose most important property can be interpreted as follows: for each irreducible finite-dimensional representation V_λ the multiplicity $m_{k\lambda,k} := \dim \operatorname{Hom}_G(V_{k\lambda}, H^0(X, L^k))$ of $V_{k\lambda}$ in $H^0(X, L^k)$ is "asymptotically given" by the volume of the convex body $\Delta_\lambda := \Delta \cap H_\lambda$. Here, λ—the so-called *highest weight*—is a parameter and $H_\lambda \subseteq \mathbb{R}^{n+1}$ is a certain affine subspace. Concretely,

$$\lim_{k\to\infty} \frac{m_{k\lambda,k}}{k^m} = \operatorname{vol}_m(\Delta_\lambda), \tag{1}$$

where m is the dimension of Δ_λ, and the volume on the right hand side denotes the m-dimensional Euclidean volume–normalized by a certain sublattice of $\mathbb{Z}^m$–of Δ_λ. In fact, this gives a Euclidean interpretation of a Duistermaat–Heckman-measure, cf. [20]. An approximation of the integral $\operatorname{vol}_m(\Delta_\lambda)$ by Riemann sums yields that the multiplicity $m_{k\lambda,k}$ is asymptotically given by the number of points of the set $\Delta_\lambda \cap \frac{1}{k}\mathbb{Z}^m$.

The construction of the body Δ is purely geometric and depends on a choice of a flag $Y_\bullet$, $Y_n \subseteq Y_{n-1} \subseteq \cdots \subseteq Y_0 = X$ of irreducible subvarieties of X, and the "successive orders of vanishing" of unipotent invariant sections $s \in H^0(X, L^k)$ along this flag. It was later realized by Kaveh and Khovanskii ([9]), and independently by Lazarsfeld and Mustaţă, ([15]), that Okounkov's construction makes sense for more general subseries of the section ring $R(X, L)$ of a line bundle over a variety X, and that the asymptotics of dimensions of linear series can be expressed as volumes of convex bodies. Specifically, the analog of (1) for the complete linear series of a big line bundle L is given by the identity

$$\lim_{k\to\infty} \frac{h^0(X, L^k)}{n!k^n} = \frac{1}{n!}\operatorname{vol}_n(\Delta_{Y_\bullet}(L)),$$

where $\Delta_{X_\bullet}(L)$ denotes the Okounkov body of the line bundle L with respect to the flag $Y_\bullet$.

The above formula shows in particular that the volume of the Okounkov body is an invariant of the line bundle L, and thus does not depend on the choice of the flag $Y_\bullet$. However, the shape of $\Delta_{X_\bullet}(L)$ depends heavily on the flag, and it is a notoriously hard problem to explicitly describe these bodies, or even to show that they possess some nice properties, such as being polyhedral. A yet more difficult problem is to determine the global Okounkov body $\Delta_{Y_\bullet}(X)$ of a variety X (cf. [15]), which is a convex cone in $\mathbb{R}^n \times N^1(X)_\mathbb{R}$ such that for each big divisor D the fibre of the second projection over $[D]$ is exactly $\Delta_{Y_\bullet}(D)$.

Returning to the original motivation by Okounkov of studying multiplicities of representations, there is also another approach to describing multiplicities by counting lattice