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Link indices of Hurwitz maps

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ABSTRACT

Let $\mathcal{M}$ be a regular map of type $\{m, n\}$, with automorphism group a smooth quotient of the ordinary $(2, m, n)$ triangle group. Then by recent work of Melekoğlu and Singerman, we can associate with $\mathcal{M}$ some positive integers called the *link indices* of $\mathcal{M}$. The number of link indices is always one, two or three, depending on the parity of m and n. In the case where m and n are both odd, which happens when $\mathcal{M}$ is a Hurwitz map (of type $\{3, 7\}$), there is a unique link index. In this paper, we prove that every positive integer divisible by 2 or 3 is the link index of some Hurwitz map. We also find the link indices of all Hurwitz maps of genus up to 10000, and of all the Hurwitz maps with automorphism group PSL$(2, q)$ for some $q < 700$, and we show that every integer between 2 and 360 is the link index of some Hurwitz map.

1. Introduction

Let X be compact Riemann surface of genus $g > 1$. Then X can be expressed in the form $\mathbb{H}/K$, where $\mathbb{H}$ denotes the hyperbolic plane and K is the fundamental group of X,

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which is a torsion free Fuchsian group, and hence a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$. An *automorphism* of X is a conformal or anti-conformal homeomorphism of X onto itself, and under composition, the set of all automorphisms of X forms a group, which we will denote by $\mathrm{Aut}^{\pm}(X)$. This group is isomorphic to Γ/K, where Γ is the normaliser of K in $\mathrm{PGL}(2,\mathbb{R})$. The subgroup of all conformal automorphisms of X is denoted by $\mathrm{Aut}^{+}(X)$, and is isomorphic Γ^{+}/K where $\Gamma^{+} = \Gamma \cap \mathrm{PSL}(2,\mathbb{R})$.

Next, a *map* $\mathcal{M}$ on the surface X is an embedding of a finite connected graph or multigraph $\mathcal{G}$ into X, such that $X \setminus \mathcal{G}$ is a union of simply connected polygonal regions, called the *faces* of $\mathcal{M}$. The *genus* of $\mathcal{M}$ is defined to be the genus of X. A directed edge is called a *dart*, and $\mathcal{M}$ is said to be *uniform*, with *type* $\{m,n\}$, if every face of $\mathcal{M}$ has m sides, and every vertex lies is incident with n darts. An *automorphism* of the map $\mathcal{M}$ is an automorphism of X that transforms $\mathcal{M}$ to itself and preserves incidence. The set of all automorphisms of $\mathcal{M}$ forms a subgroup of $\mathrm{Aut}^{\pm}(X)$, which we will denote by $\mathrm{Aut}^{\pm}(\mathcal{M})$, and similarly, the subgroup of all conformal automorphisms of $\mathcal{M}$ by $\mathrm{Aut}^{+}(\mathcal{M})$.

If $\mathrm{Aut}^{+}(\mathcal{M})$ is transitive on the darts of $\mathcal{M}$, then $\mathcal{M}$ is a *regular* map. In that case, $\mathcal{M}$ is uniform, and if its type is $\{m,n\}$ then its fundamental group K is normal in a Fuchsian group isomorphic to the *ordinary triangle group* $\Gamma[2,m,n]$, which is the abstract group with presentation

$$\langle\, x, y, z \mid x^2 = y^m = z^n = xyz = 1 \,\rangle, \tag{1.1}$$

and the factor group $\Gamma[2,m,n]/K$ is isomorphic to $\mathrm{Aut}^{+}(\mathcal{M})$. See [11] for further details.

Next, if $\mathcal{M}$ admits an involutory anti-conformal automorphism σ that preserves an edge and interchanges the two darts associated with that edge without interchanging the two incident faces, then the map $\mathcal{M}$ is called *reflexible*, and σ is called a *reflection* of $\mathcal{M}$. If $\mathcal{M}$ is not reflexible, then it is called *chiral*. The fixed point set of any reflection σ of $\mathcal{M}$ consists of k simple closed geodesics on X, which are called the *mirrors* of σ. By a theorem of Harnack [8], it is well known that $k \le g+1$, where g is the genus of X.

If $\mathcal{M}$ is regular and reflexible, then its fundamental group K is also normal in an isomorphic copy of the *extended triangle group* $\Gamma(2,m,n)$, which has abstract presentation

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^m = (ca)^n = 1 \,\rangle. \tag{1.2}$$

In this case the factor group $\Gamma(2,m,n)/K$ is isomorphic to $\mathrm{Aut}^{\pm}(\mathcal{M})$. Note here that the subgroup generated by $x = ab$, $y = bc$ and $z = ca$ is isomorphic to $\Gamma[2,m,n]$.

A classical theorem of Hurwitz [9] states that a compact Riemann surface of genus $g > 1$ has at most $84(g-1)$ conformal automorphisms. Any such surface $X = \mathbb{H}/K$ is called a *Hurwitz surface*, and in that case $\mathrm{Aut}^{+}(X)$ is called a *Hurwitz group*. Furthermore, it is known that if X is a Hurwitz surface then its fundamental group K is normal in a copy of the ordinary triangle group $\Gamma[2,3,7]$, and so every Hurwitz surface underlies a regular map of type $\{3,7\}$, often called a *Hurwitz map*. The upper bound in Hurwitz's theorem is attained for infinitely many values of the genus g, and hence there exist infinitely many Hurwitz maps and surfaces; see [13] and also [4,5].