Journal of Algebra 486 (2017) 36–63

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Group actions on algebraic stacks via butterflies

Behrang Noohi

A R T I C L E I N F O

Article history:
Received 21 April 2016
Available online 5 May 2017
Communicated by Paul Balmer

Keywords:
Algebraic stacks
Actions on stacks
2-groups
Categorical groups
Weighted projective stacks
Weighted projective general linear groups

A B S T R A C T

We introduce an explicit method for studying actions of a group stack $\mathcal{G}$ on an algebraic stack $\mathcal{X}$. As an example, we study in detail the case where $\mathcal{X} = \mathcal{P}(n_0, \cdots, n_r)$ is a weighted projective stack over an arbitrary base S. To this end, we give an explicit description of the group stack of automorphisms of $\mathcal{P}(n_0, \cdots, n_r)$, the *weighted projective general linear 2-group* $\mathrm{PGL}(n_0, \cdots, n_r)$. As an application, we use a result of Colliot-Thélène to show that for every linear algebraic group G over an arbitrary base field k (assumed to be reductive if $\mathrm{char}(k) > 0$) such that $\mathrm{Pic}(G) = 0$, every action of G on $\mathcal{P}(n_0, \cdots, n_r)$ lifts to a linear action of G on $\mathbb{A}^{r+1}$.

Contents

E-mail address: b.noohi@qmul.ac.uk.

http://dx.doi.org/10.1016/j.jalgebra.2017.05.002

1. Introduction

The aim of this work is to propose a concrete method for studying group actions on algebraic stacks. Of course, in its full generality this problem could already be very difficult in the case of schemes. The case of stacks has yet an additional layer of difficulty due to the fact that stacks have two types of symmetries: 1-symmetries (i.e., self-equivalences) and 2-symmetries (i.e., 2-morphisms between self-equivalences).

Studying actions of a group stack $\mathcal{G}$ on a stack $\mathcal{X}$ can be divided into two subproblems. One, which is of geometric nature, is to understand the two types of symmetries alluded to above; these can be packaged in a group stack $\mathcal{A}ut\mathcal{X}$. The other, which is of homotopy theoretic nature, is to get a hold of morphisms $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$. Here, a morphism $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ means a weak monoidal functor; two morphisms $f, g\colon \mathcal{G} \to \mathcal{A}ut\mathcal{X}$ that are related by a monoidal transformation $\varphi\colon f \Rightarrow g$ should be regarded as giving rise to the "same" action.

Therefore, to study actions of $\mathcal{G}$ on $\mathcal{X}$ one needs to understand the group stack $\mathcal{A}ut\mathcal{X}$, the morphisms $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$, and also the transformations between such morphisms. Our proposed method, uses techniques from 2-group theory to tackle these problems. It consists of two steps:

1) finding suitable crossed module models for $\mathcal{A}ut\mathcal{X}$ and $\mathcal{G}$;
2) using butterflies [11,1] to give a geometric description of morphisms $\mathcal{G} \to \mathcal{A}ut\mathcal{X}$ and monoidal transformations between them.

Finding a 'suitable' crossed module model for $\mathcal{A}ut\mathcal{X}$ may not always be easy, but we can go about it by choosing a suitable 'symmetric enough' atlas $X \to \mathcal{X}$. This can be used to find an approximation of $\mathcal{A}ut\mathcal{X}$ (Proposition 6.2), and if we are lucky (e.g., when $\mathcal{X} = \mathcal{P}(n_0, \cdots, n_r)$) it gives us the whole $\mathcal{A}ut\mathcal{X}$.

Once crossed module models for $\mathcal{G}$ and $\mathcal{A}ut\mathcal{X}$ are found, the butterfly method reduces the action problem to standard problems about group homomorphisms and group extensions, which can be tackled using techniques from group theory.

Organization of the paper

Sections §3–§5 are devoted to setting up the basic homotopy theory of 2-group actions and using butterflies to formulate our strategy for studying actions. To illustrate our method, in the subsequent sections we apply these ideas to study group actions on weighted projective stacks. In §6 we define weighted projective general linear 2-groups