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Global parameter test ideals

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ABSTRACT

This paper shows the existence of ideals whose localizations and completions at prime ideals are parameter test ideals of the localized and completed rings. We do this for Cohen–Macaulay localizations (resp., completions) of non-local rings, for generalized Cohen–Macaulay rings, and for non-local rings with isolated non-Cohen–Macaulay points, each being an isolated non-F-rational point. The tools used to prove these results are constructive in nature and as a consequence our results yield algorithms for the computation of these global parameter test ideals.
Finally, we illustrate the power of our methods by analyzing the HSL numbers of local cohomology modules with support at any prime ideal.

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1. Introduction

This paper studies certain properties of commutative rings of prime characteristic p. Such rings A are equipped with *Frobenius maps* $f_e : A \to A$ defined as $f_e(a) = a^{p^e}$ and these give a good handle on various problems which are not available in characteristic zero. One such handle is provided by the machinery of *tight closure* introduced by Mel Hochster and Craig Huneke in the 1990's (cf. [7]) which we now review.

The tight closure of an ideal $I \subseteq A$ is the set of all $a \in A$ for which for some element $c \in A$ not in any minimal prime one has $ca^{p^e} \in I^{[p^e]}$ for all $e \gg 0$, where $I^{[p^e]}$ denotes the ideal of A generated by all p^eth powers of elements in I.

One of the basic properties of tight closure is that for ideals I of regular rings one has $I^* = I$ (we refer to ideals with this property as being *tightly closed*), or equivalently, that $c = 1$ and $e \geq 0$ can be used in the definition above to test membership in the tight closure of ideals. This suggests that measuring the failure of the tight closure operation to be trivial might be a useful way of measuring how bad the singularity of a ring is, and one thus obtains a hierarchy of singularities:

(a) regular rings,
(b) *F-regular* rings: all ideals in all localizations are tightly closed,
(c) *weakly F-regular* rings: all ideals are tightly closed,
(d) *F-rational* rings: all ideals generated by parameters are tightly closed.

From this point of view, the object of interest is the set of elements c in the definition of tight closure which can be used to test membership in the tight closure of ideals.

Definition 1.1. (cf. [7, §6] and [9, §8]) An element c not in any minimal prime is a *test element* if for all ideals I and all $e \geq 0$, $cI^{*[p^e]} \subseteq I^{[p^e]}$. The *test ideal* is defined as the ideal generated by all test elements.

An element c not in any minimal prime is a *parameter test element* if for all ideals I generated by parameters and all $e \geq 0$, $cI^{*[p^e]} \subseteq I^{[p^e]}$. The *parameter test ideal* is the ideal generated by all parameter test elements.

Thus properties (c) and (d) above can be restated as the test ideal and the parameter test ideal being the unit ideal, respectively.

In this paper we construct *global parameter test ideals* of finitely generated algebras, i.e., ideals whose localization are parameter test ideals of the localized rings. Among other things, our explicit description of these ideals yields an explicit description of the F-rational locus of finitely generated algebras, recovering the fact that the F-rational locus is open (cf. [23]), and in the process also providing a method for computing global parameter test ideals.

2. Prime characteristic tools

In this section we introduce various tools and notation used to study rings of prime characteristic and their modules.