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Growth of Hilbert coefficients of Syzygy modules

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ABSTRACT

Let $(A, \mathfrak{m})$ be a local complete intersection ring of dimension d and let I be an $\mathfrak{m}$-primary ideal. Let M be a maximal Cohen–Macaulay A-module. For $i = 0, 1, \cdots, d$, let $e_i^I(M)$ denote the ith Hilbert-coefficient of M with respect to I. We prove that for $i = 0, 1, 2$, the function $j \mapsto e_i^I(\operatorname{Syz}_j^A(M))$ is of quasi-polynomial type with period 2. Let $G_I(M)$ be the associated graded module of M with respect to I. If $G_I(A)$ is Cohen–Macaulay and $\dim A \leq 2$ we also prove that the functions $j \mapsto \operatorname{depth} G_I(\operatorname{Syz}_{2j+i}^A(M))$ are eventually constant for $i = 0, 1$. Let $\xi_I(M) = \lim_{l \to \infty} \operatorname{depth} G_{I^l}(M)$. Finally we prove that if $\dim A = 2$ and $G_I(A)$ is Cohen–Macaulay then the functions $j \mapsto \xi_I(\operatorname{Syz}_{2j+i}^A(M))$ are eventually constant for $i = 0, 1$.

1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d and let M be a finitely generated A-module of dimension r. Let I be an $\mathfrak{m}$-primary ideal. Let $\ell(N)$ denote the length of an A-module N. The function $H_I^{(1)}(M, n) = \ell(M/I^{n+1}M)$ is called the *Hilbert–Samuel* function of M with respect to I. It is well-known that there exists a polynomial $P_I(M, X) \in \mathbb{Q}[X]$ of degree r such that $P_I(M, n) = H_I^{(1)}(M, n)$ for $n \gg 0$. The poly-

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nomial $P_I(M,X)$ is called the Hilbert–Samuel polynomial of M with respect to I. We write

$$P_I(M,X) = \sum_{i=0}^{r}(-1)^i e_i^I(M)\binom{X+r-i}{r-i}.$$

The integers $e_i^I(M)$ are called the ith-Hilbert coefficient of M with respect to I. The zeroth Hilbert coefficient $e_0^I(M)$ is called the *multiplicity* of M with respect to I.

For $j \geq 0$ let $\operatorname{Syz}_j^A(M)$ denote the jth syzygy of M. In this paper we investigate the function $j \mapsto e_i^I(\operatorname{Syz}_j^A(M))$ for $i \geq 0$. It becomes quickly apparent that for reasonable answers we need that the minimal resolution of M should have some structure. Minimal resolutions of modules over complete intersection rings have a good structure. If $A = B/(f_1, \cdots, f_c)$ with $\mathbf{f} = f_1, \cdots, f_c$ a B-regular sequence and $\operatorname{projdim}_B M$ is finite then also the minimal resolution of M has a nice structure. The definitive class of modules with a good structure theory of their minimal resolution is the class of modules with finite complete intersection dimension, see [2]. We are able to prove our results for a more restrictive class of modules than modules of finite CI-dimension.

Definition 1.1. We say the A module M has finite GCI-dimension if there is a flat local extension $(B, \mathfrak{n})$ of A such that

(1) $\mathfrak{m}B = \mathfrak{n}$.
(2) $B = Q/(f_1, \cdots, f_c)$, where Q is local and $f_1, \ldots, f_c$ is a Q-regular sequence.
(3) $\operatorname{projdim}_Q M \otimes_A B$ is finite.

We note that every finitely generated module over an abstract complete intersection ring has finite GCI dimension. If $A = R/(f_1, \cdots, f_c)$ with $f_1, \cdots, f_c$ a R-regular sequence and $\operatorname{projdim}_R M$ is finite then also M has finite GCI-dimension. We also note that if M has finite GCI dimension then it has finite CI-dimension. Although our notion of GCI dimension is weaker than the notion of CI-dimension, it should be noted that all known examples of modules having finite CI-dimension also has finite GCI-dimension. See 2.19 for reasons why we do not study modules with finite CI-dimension (the general case).

If M has finite CI-dimension then the function $i \mapsto \ell(\operatorname{Tor}_i^A(M,k))$ is of quasi-polynomial type with degree two. Set $\operatorname{cx}(M)$ = degree of this function $+1$. (See 2.7 for degree of a function of quasi-polynomial type).

Let $G_I(A) = \bigoplus_{n\geq 0} I^n/I^{n+1}$ be the associated graded ring of A with respect to I. Let $G_I(M) = \bigoplus_{n\geq 0} I^nM/I^{n+1}M$ be the associated graded module of M with respect to I. Our main result is

Theorem 1.2. *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension d and let M be a maximal Cohen–Macaulay A-module. Let I be an $\mathfrak{m}$-primary ideal. Assume M has finite GCI dimension. Then for $i = 0, 1, 2$, the function $j \mapsto e_i^I(\operatorname{Syz}_j^A(M))$ is of quasi-polynomial type with period two and degree $\leq \operatorname{cx}(M) - 1$.*