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Abstract

In this paper, we characterize Ulrich modules over cyclic quotient surface singularities using the notion of special Cohen-Macaulay modules. We also investigate the number of indecomposable Ulrich modules for a given cyclic quotient surface singularity, and show that the number of exceptional curves in the minimal resolution determines a boundary on the number of indecomposable Ulrich modules.

Keywords: Ulrich modules, special Cohen-Macaulay modules, McKay correspondence, cyclic quotient surface singularities
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1. Introduction

Let $(R, \mathfrak{m}, \Bbbk)$ be a Cohen-Macaulay (= CM) local ring, with $\dim R = d$. For a finitely generated R-module M, we say that M is a maximal Cohen-Macaulay (= MCM) R-module if $\operatorname{depth}_R M = d$. For each MCM R-module M, we have that $\mu_R(M) \le \mathrm{e}_{\mathfrak{m}}(M)$, where $\mu_R(M)$ denotes the number of minimal generators (i.e., $\mu_R(M) = \dim_{\Bbbk} M/\mathfrak{m}M$), and $\mathrm{e}_{\mathfrak{m}}(M)$ is the multiplicity of M with respect to $\mathfrak{m}$. Note that if R is a domain, then we have that $\mathrm{e}_{\mathfrak{m}}(M) = (\operatorname{rank}_R M)\mathrm{e}_{\mathfrak{m}}(R)$.

An Ulrich module is defined as a module that has the maximum number of generators with respect to the above inequality. We sometimes call this a maximally generated maximal Cohen-Macaulay module, in line with the original terminology [Ulr, BHU]. The name "Ulrich modules" was introduced in [HK]. We remark that the conditions below are inherited by direct summands and direct sums, and hence Ulrich modules are closed under direct summands and direct sums.

Definition 1.1 ([Ulr, BHU])**.** Let M be an MCM R-module. We say that M is an Ulrich module if it satisfies $\mu_R(M) = \mathrm{e}_{\mathfrak{m}}(M)$.

Several properties of these modules have been investigated in the aforementioned references. In a more geometric setting, they have been studied as Ulrich bundles, for example in [ESW, CH1, CH2, CKM]. Recently, this notion was generalized for each non-parameter $\mathfrak{m}$-primary ideal I in [GOTWY1], and this notion has been actively studied (cf. [GOTWY2, GOTWY3]). Namely, we say that an MCM R-module M is an Ulrich module "with respect to I" if it satisfies the following conditions:

$$(1)\, \mathrm{e}_I(M) = \ell_R(M/IM), \quad (2)\, M/IM \text{ is an } R/I\text{-free module},$$

where $\mathrm{e}_I(M)$ is the multiplicity of M with respect to I, and $\ell_R(M/IM)$ denotes the length of M/IM. Thus, an Ulrich module with respect to $\mathfrak{m}$ is nothing else but an Ulrich module in the sense of Definition 1.1. (The condition (2) is automatically satisfied if $I = \mathfrak{m}$.) In addition, Ulrich modules have appeared in an attempt to formulate the notion of "almost Gorenstein rings" [GTT]. Thus, it has become more important to understand these modules. However, even the existence of an Ulrich module for a given CM local ring is still not known in general. Another important problem is to characterize (and classify) Ulrich modules when a given ring R admits an Ulrich module. For example, we know the existence of such a module for the following cases:

- A two dimensional domain with an infinite field [BHU].

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