Journal of Algebra 482 (2017) 303–345

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The bounded derived categories of an algebra with radical squared zero ☆

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ARTICLE INFO

Article history:
Received 19 October 2016
Available online 23 April 2017
Communicated by Luchezar L. Avramov

MSC:
16E35
16G20
16G70
18E30

Keywords:
Representations of a quiver
Modules over a linear category
Triangulated categories
Derived categories
Irreducible morphisms
Almost split sequences
Almost split triangle
Auslander–Reiten quiver
Galois covering

ABSTRACT

Let Λ be an elementary locally bounded linear category over a field with radical squared zero. We shall show that the bounded derived category $D^b(\mathrm{Mod}^b\Lambda)$ of finitely supported left Λ-modules admits a Galois covering which is the bounded derived category of almost finitely co-presented representations of a gradable quiver. Restricting to the bounded derived category $D^b(\mathrm{mod}^b\Lambda)$ of finite dimensional left Λ-modules, we shall be able to describe its indecomposable objects, obtain a complete description of the shapes of its Auslander–Reiten components, and classify those Λ such that $D^b(\mathrm{mod}^b\Lambda)$ has only finitely many Auslander–Reiten components.

☆ The first named author is supported in part by Fondo Sectorial SEP-CONACYT via the project 43374F, while the second named author is supported in part by the Natural Sciences and Engineering Research Council of Canada, OGP172797.

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http://dx.doi.org/10.1016/j.jalgebra.2017.04.006
0021-8693/

Introduction

Throughout this paper, k denotes a commutative field. One of the central topics in the representation theory of a finite dimensional k-algebra A is to study its bounded derived category $D^b(\mathrm{mod}A)$ of finitely generated left modules; see, for example, [17, 18]. Indeed, the triangulated category $D^b(\mathrm{mod}A)$ captures all the homological properties of the algebra A. Since it is Hom-finite and Krull–Schmidt, we are particularly interested in classifying the indecomposable objects and studying the Auslander–Reiten theory of irreducible morphisms and almost split triangles in $D^b(\mathrm{mod}A)$. These objectives have been achieved to a certain extent in the hereditary case; see [8,17,18]. In case A is a gentle algebra, the indecomposable objects of $D^b(\mathrm{mod}\,A)$ have been explicitly described in [11]. Speaking of the Auslander–Reiten theory, it has been shown that an indecomposable complex of $D^b(\mathrm{mod}\,A)$ is the ending (respectively, starting) term of an almost split triangle if and only if it is isomorphic to a bounded complex of finitely generated projective (respectively, injective) A-modules; and consequently, the Auslander–Reiten quiver of $D^b(\mathrm{mod}A)$ is stable if and only if A is of finite global dimension; see [17,19]. In a general setting, some particular types of stable Auslander–Reiten components of $D^b(\mathrm{mod}A)$ are investigated in [27]. In case A is self-injective with no simple block, the stable Auslander–Reiten components of $D^b(\mathrm{mod}\,A)$ are of shape $\mathbb{Z}\mathbb{A}_\infty$; see [29, (3.7)], whereas the non-stable ones occur rarely and are explicitly described in [20, (5.7)].

The objective of this paper is to deal with algebras with radical squared zero. Our main tool is the covering technique, which was introduced in [12,14,16] and further developed in [2,3,7]. This requires us to work in a more general setting. Indeed, let Λ be a connected locally bounded k-category with radical squared zero. We shall assume that Λ is *elementary*, that is, all simple left Λ-modules are one dimensional over k. By Gabriel's theorem, $\Lambda \cong kQ/(kQ^+)^2$, where Q is a connected locally finite quiver called the *ordinary quiver* of Λ, and kQ is the path category of Q over k with an ideal kQ^+ generated by the arrows; see [12, (2.2)]. Our aim is to study the bounded derived category $D^b(\mathrm{Mod}^b\Lambda)$ of finitely supported left Λ-modules and the bounded derived category $D^b(\mathrm{mod}^b\Lambda)$ of finite dimensional left Λ-modules. Observe that if Λ is a finite dimensional k-algebra, then $D^b(\mathrm{Mod}^b\Lambda)$ is the bounded derived category of all left Λ-modules.

In case Q is gradable, using of the Koszul functor considered in [9], we shall obtain a triangle-equivalent $\mathscr{F} : D^b(\mathrm{Rep}^-(Q^{\mathrm{op}})) \to D^b(\mathrm{Mod}^b\Lambda)$, called *Koszul equivalence*; see (3.9), where $\mathrm{Rep}^-(Q^{\mathrm{op}})$ is the hereditary abelian category of almost finitely copresented representations of the opposite quiver Q^{op} of Q; see [8, (1.8)]. In the general case, we shall choose a minimal gradable covering $\pi : \tilde{Q} \to Q$, which induces a Galois covering $\pi_\lambda^D : D^b(\mathrm{Mod}^b\tilde{\Lambda}) \to D^b(\mathrm{Mod}^b\Lambda)$, where $\tilde{\Lambda} = k\tilde{Q}/(k\tilde{Q}^+)^2$; see [7, (7.10)]. Composing this covering with the Koszul equivalence, we shall obtain a Galois covering $\mathfrak{F}_\pi : D^b(\mathrm{Rep}^-(\tilde{Q}^{\,\mathrm{op}})) \to D^b(\mathrm{Mod}^b\Lambda)$; see (4.10).

Restricting to finite dimensional modules, we shall obtain a Galois covering $\mathfrak{F}_\pi : D^b(\mathrm{rep}^-(\tilde{Q}^{\mathrm{op}})) \to D^b(\mathrm{mod}^b\Lambda)$, where $\mathrm{rep}^-(\tilde{Q}^{\mathrm{op}})$ is the hereditary abelian category of