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Noether resolutions in dimension 2

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ABSTRACT

Let $R := K[x_1, \ldots, x_n]$ be a polynomial ring over an infinite field K, and let $I \subset R$ be a homogeneous ideal with respect to a weight vector $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{Z}^+)^n$ such that $\dim(R/I) = d$. In this paper we study the minimal graded free resolution of R/I as A-module, that we call the Noether resolution of R/I, whenever $A := K[x_{n-d+1}, \ldots, x_n]$ is a Noether normalization of R/I. When $d = 2$ and I is saturated, we give an algorithm for obtaining this resolution that involves the computation of a minimal Gröbner basis of I with respect to the weighted degree reverse lexicographic order. In the particular case when R/I is a 2-dimensional semigroup ring, we also describe the multigraded version of this resolution in terms of the underlying semigroup. Whenever we have the Noether resolution of R/I or its multigraded version, we obtain formulas for the corresponding Hilbert series of R/I, and when I is homogeneous, we obtain a formula for the Castelnuovo–Mumford regularity of R/I. Moreover, in the more general setting that R/I is a simplicial semigroup ring of any dimension, we provide its Macaulayfication.

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As an application of the results for 2-dimensional semigroup rings, we provide a new upper bound for the Castelnuovo–Mumford regularity of the coordinate ring of a projective monomial curve. Finally, we describe the multigraded Noether resolution and the Macaulayfication of either the coordinate ring of a projective monomial curve $\mathcal{C} \subseteq \mathbb{P}^n_K$ associated to an arithmetic sequence or the coordinate ring of any canonical projection $\pi_r(\mathcal{C})$ of $\mathcal{C}$ to $\mathbb{P}^{n-1}_K$.

1. Introduction

Let $R := K[x_1, \ldots, x_n]$ be a polynomial ring over an infinite field K, and let $I \subset R$ be a weighted homogeneous ideal with respect to the vector $\omega = (\omega_1, \ldots, \omega_n) \in (\mathbb{Z}^+)^n$, i.e., I is homogeneous for the grading $\deg_\omega(x_i) = \omega_i$. We denote by d the Krull dimension of R/I and we assume that $d \geq 1$. Suppose $A := K[x_{n-d+1}, \ldots, x_n]$ is a Noether normalization of R/I, i.e., $A \hookrightarrow R/I$ is an integral ring extension. Under this assumption R/I is a finitely generated A-module, so to study the minimal graded free resolution of R/I as A-module is an interesting problem. Set

$$\mathcal{F} : 0 \longrightarrow \oplus_{v \in \mathcal{B}_p} A(-s_{p,v}) \xrightarrow{\psi_p} \cdots \xrightarrow{\psi_1} \oplus_{v \in \mathcal{B}_0} A(-s_{0,v}) \xrightarrow{\psi_0} R/I \longrightarrow 0$$

this resolution, where for all $i \in \{0, \ldots, p\}$ $\mathcal{B}_i$ denotes some finite set, and $s_{i,v}$ are nonnegative integers. This work concerns the study of this resolution of R/I, which will be called the *Noether resolution of* R/I. More precisely, we aim at determining the sets $\mathcal{B}_i$, the shifts $s_{i,v}$ and the morphisms ψ_i.

One of the characteristics of Noether resolutions is that they have shorter length than the minimal graded free resolution of R/I as R-module. Indeed, the projective dimension of R/I as A-module is $p = d - \operatorname{depth}(R/I)$, meanwhile its projective dimension of R/I as R-module is $n - \operatorname{depth}(R/I)$. Studying Noether resolutions is interesting since they contain valuable information about R/I. For instance, since the Hilbert series is an additive function, we get the Hilbert series of R/I from its Noether resolution. Moreover, whenever I is a homogeneous ideal, i.e., homogeneous for the weight vector $\omega = (1, \ldots, 1)$, one can obtain the Castelnuovo–Mumford regularity of R/I in terms of the Noether resolution as $\operatorname{reg}(R/I) = \max\{s_{i,v} - i \mid 0 \leq i \leq p, v \in \mathcal{B}_i\}$.

In Section 2 we start by describing in Proposition 1 the first step of the Noether resolution of R/I. By Auslander–Buchsbaum formula, the depth of R/I equals $d - p$. Hence, R/I is Cohen–Macaulay if and only if $p = 0$ or, equivalently, if R/I is a free A-module. This observation together with Proposition 1, lead to Proposition 2 which is an effective criterion for determining whether R/I is Cohen–Macaulay or not. This criterion generalizes [4, Proposition 2.1]. If R/I is Cohen–Macaulay, Proposition 1 provides the whole Noether resolution of R/I. When $d = 1$ and R/I is not Cohen–Macaulay, we describe the Noether resolution of R/I by means of Proposition 1 together with Proposition 3.