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Congruence subgroups from representations of the three-strand braid group ☆

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ABSTRACT

Ng and Schauenburg proved that the kernel of a $(2+1)$-dimensional topological quantum field theory representation of $\mathrm{SL}(2,\mathbb{Z})$ is a congruence subgroup. Motivated by their result, we explore when the kernel of an irreducible representation of the braid group B_3 with finite image enjoys a congruence subgroup property. In particular, we show that in dimensions two and three, when the projective order of the image of the braid generator σ_1 is between 2 and 5 the kernel projects onto a congruence subgroup of $\mathrm{PSL}(2,\mathbb{Z})$ and compute its level. However, we prove that for three dimensional representations, the projective order is not enough to decide the congruence property. For each integer of the form $2\ell \geq 6$ with ℓ odd, we construct a pair of non-congruence subgroups associated with three-dimensional representations having finite image and σ_1 mapping to a matrix with projective order 2ℓ. Our technique uses classification results of low dimensional braid group representations, and the Fricke–Wohlfahrt theorem in number theory.

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1. Introduction

The double cover $\mathrm{SL}(2,\mathbb{Z})$ of the modular group $\mathrm{PSL}(2,\mathbb{Z})$ naturally occurs in quantum topology as the mapping class group of the torus. Let $\Sigma_{g,n}$ be the orientable genus g surface with n punctures and denote by $\mathrm{Mod}(\Sigma_{g,n})$ its mapping class group. A (2+1)-dimensional topological quantum field theory (TQFT) affords a canonical projective representation of $\mathrm{Mod}(\Sigma_{g,n})$ which we refer to as a **quantum representation**. An amazing theorem of Ng and Schauenburg [9] says that the kernel of the quantum representations of $\mathrm{SL}(2,\mathbb{Z})$ is always a congruence subgroup. The modular group is also disguised as the three-strand braid group B_3 through the central extension: $1 \to \mathbb{Z} = \langle(\sigma_1\sigma_2)^3\rangle \to B_3 \to \mathrm{PSL}(2,\mathbb{Z}) \to 1$. Each simple object of the modular tensor category $\mathcal{C}$ associated to a $(2+1)$-TQFT gives rise to a representation of B_3. Are there versions of the Ng–Schauenburg congruence kernel theorem for those braid group representations? We initiate a systematic investigation of this problem and find that a native generalization does not hold.

To pass from a representation of B_3 to the modular group $\mathrm{PSL}(2,\mathbb{Z})$, we consider only irreducible representations $\rho_x : B_3 \to \mathrm{GL}(d,\mathbb{C})$ associated to a simple object x of a modular tensor category $\mathcal{C}$. Then the generator $(\sigma_1\sigma_2)^3$ of the center of B_3 is a scalar of finite order. By rescaling ρ_x with a root of unity ξ, we obtain a representation of the modular group $\rho_{x,\xi} : \mathrm{PSL}(2,\mathbb{Z}) \to \mathrm{GL}(d,\mathbb{C})$. By the property F conjecture, the representations $\rho_{x,\xi}$ should have finite images if the squared quantum dimension d_x^2 of x is an integer. For the Ising anyon σ, the kernel is indeed a congruence subgroup, but the kernel for the anyon denoted as G in $D(S_3)$ is not [3]. Therefore, when a property F anyon has a congruence subgroup property is more subtle. In this paper we systematically explore the low dimensional irreducible representations of B_3 with finite images, and determine when the kernel is a congruence subgroup. Some basic results on finite quotient of the braid groups include [4,12,2].

Congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$ are well-studied as they are easy examples of finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ and because of their role in the theory of modular forms and functions. Their relative scarcity in the collection of all finite index subgroups of $\mathrm{SL}(2,\mathbb{Z})$ makes them of interest. Indeed, if we let $N_c(n)$ (resp. $N(n)$) denote the number of congruence subgroups (resp. subgroups) of $\mathrm{SL}(2,\mathbb{Z})$ with index n then $N_c(n)/N(n) \to 0$ as $n \to \infty$ [11]. Contrast this with the result of Bass, Lazard, and Serre [1] and separately Mennicke [8] stating that for d greater than two any finite index subgroup of $\mathrm{SL}(d,\mathbb{Z})$ is a congruence subgroup.

Another motivation of this research is to study the vector-valued modular forms (VVMF) associated to congruence subgroups (see [6] and the references therein). VVMFs provide deep insight for the study of TQFTs and conformal field theories (CFTs). Since the general VVMF theory applies also to non-congruence subgroups, TQFT representations of B_3 provide interesting test ground of the theory and conversely, VVMF could provide deep insight into the study of the TQFT representations of B_3 even in the non-congruence case. The matrices contained in the image of a B_3 representation can also be