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Weakly Cohen–Macaulay posets and a class of finite-dimensional graded quadratic algebras

Tyler Kloefkorn

Department of Mathematics, University of Arizona, Tucson, AZ 85721, United States

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ABSTRACT

To a finite ranked poset Γ we associate a finite-dimensional graded quadratic algebra R_Γ. Assuming Γ satisfies a combinatorial condition known as uniform, R_Γ is related to a well-known algebra, the *splitting algebra* A_Γ. First introduced by Gelfand, Retakh, Serconek, and Wilson, splitting algebras originated from the problem of factoring non-commuting polynomials. Given a finite ranked poset Γ, we ask: Is R_Γ Koszul? The Koszulity of R_Γ is related to a combinatorial topology property of Γ called Cohen–Macaulay. Kloefkorn and Shelton proved that if Γ is a finite ranked cyclic poset, then Γ is Cohen–Macaulay if and only if Γ is uniform and R_Γ is Koszul. We define a new generalization of Cohen–Macaulay, weakly Cohen–Macaulay. This new class includes non-uniform posets and posets with disconnected open subintervals. Using a spectral sequence associated to Γ and the notion of a noncommutative Koszul filtration for R_Γ, we prove: if Γ is a finite ranked cyclic poset, then Γ is weakly Cohen–Macaulay if and only if R_Γ is Koszul. In addition, we prove that Γ is Cohen–Macaulay if and only if Γ is uniform and weakly Cohen–Macaulay.

E-mail address: tkloefkorn@math.arizona.edu.

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1. Introduction

We fix a field $\mathbb{F}$. Let Γ denote a finite ranked poset with unique minimal element $*$ and order $<$. If $x, y \in \Gamma$, we write $x \to y$ if x covers y.

Definition 1.1. The algebra R_Γ is the graded quadratic $\mathbb{F}$-algebra with degree one generators r_x for all $x \in \Gamma \setminus \{*\}$ and relations

$$r_x \sum_{x \to y} r_y = 0 \qquad \text{and} \qquad r_x r_w = 0 \text{ whenever } x \nrightarrow w.$$

The algebra R_Γ has a relatively complex history – we give a brief summary. In [3], Gelfand, Retakh, Serconek and Wilson associate to Γ a connected graded $\mathbb{F}$-algebra A_Γ, which is called the *splitting algebra* of Γ; splitting algebras are related to the problem of factoring non-commuting polynomials. Retakh, Serconek and Wilson later showed that if Γ satisfies a combinatorial condition called *uniform*, then an associated graded algebra of the splitting algebra is quadratic, and it follows that A_Γ is quadratic (cf. [8]). These authors then asked a standard question in homological algebra: given a finite uniform ranked Γ, is A_Γ Koszul?

Definition 1.2. A connected graded $\mathbb{F}$-algebra A is *Koszul* if the trivial right A-module $\mathbb{F}_A$ admits a linear projective resolution.

We note that Koszul algebras are quadratic and that there are many equivalent definitions (cf. [7]).

If A_Γ is Koszul, one can use the Hilbert series condition of numerical Koszulity to extract combinatorial data from the algebra. Recent work in the area of splitting algebras often focuses on calculating Hilbert series (cf. [9] and [10]).

Given a specific Γ, it is difficult to determine if A_Γ is Koszul. In fact, preliminary literature incorrectly asserted that A_Γ is Koszul for all uniform Γ. We thus pass to a related question. Following [8] and assuming Γ is uniform, we filter A_Γ by rank in Γ. We denote the associated graded algebra by grA_Γ. Finally, we study the quadratic dual of grA_Γ, which is denoted by $(grA_\Gamma)^!$. Applying standard techniques, we know that if $(grA_\Gamma)^!$ is Koszul, then so is A_Γ. We then ask: given a finite uniform ranked Γ, is $(grA_\Gamma)^!$ Koszul? We note in [9] and subsequent papers by the same authors, $(grA_\Gamma)^!$ is denoted by $B(\Gamma)$.

If Γ is uniform, then $R_\Gamma = (grA_\Gamma)^!$. The notation R_Γ is from [2]; Cassidy, Phan and Shelton assume Γ is uniform and denote $(grA_\Gamma)^!$ with R_Γ. We emphasize: for Definition 1.1, Γ need not be uniform.

We are interested in the algebra R_Γ and Koszulity of R_Γ, even if Γ is not uniform and we draw no conclusions about splitting algebras. In [2], Cassidy, Phan and Shelton show that there exists a non-Koszul R_Γ. Also, if Γ stems from a geometric object, then the Koszulity of R_Γ gives important combinatorial and topological data; for example,