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Étale representations for reductive algebraic groups with one-dimensional center

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ABSTRACT

A complex vector space V is a prehomogeneous G-module if G acts rationally on V with a Zariski-open orbit. The module is called étale if $\dim V = \dim G$. We study étale modules for reductive algebraic groups G with one-dimensional center. For such G, even though every étale module is a regular prehomogeneous module, its irreducible submodules have to be non-regular. For these non-regular prehomogeneous modules, we determine some strong constraints on the ranks of their simple factors. This allows us to show that there do not exist étale modules for $G = \mathrm{GL}_1 \times S \times \cdots \times S$, with S simple.

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Contents

1. Introduction

Affine étale representations of Lie groups arise in many contexts. For a given connected Lie group G, the existence of such a representation is equivalent to the existence of a left-invariant affine structure on G (see [2,4]). In 1977 Milnor [15] discussed the importance of such structures for the study of fundamental groups of complete affine manifolds, and for the study of affine crystallographic groups, which initiated generalizations of the Bieberbach theorems for Euclidean crystallographic groups to affine crystallographic groups, see [10]. Milnor asked the existence question for left-invariant affine structures on a given Lie group G, and suggested that all solvable Lie groups G admit such a structure. This question received a lot of attention, and was eventually answered negatively by Benoist [3]. For a survey on the results and the history see [6,7, 10].

Affine étale representations of G and left-invariant affine structures on G both define a bilinear product on the Lie algebra $\mathfrak{g}$ of G that gives $\mathfrak{g}$ the structure of a *left-symmetric algebra* (*LSA-structure* for short), and conversely an LSA-structure determines an affine structure on G. The existence question then can be formulated on the Lie algebra level, and has been studied for several classes of Lie algebras, e.g., for semisimple, reductive, nilpotent and solvable Lie algebras, see [6]. LSA-structures on Lie algebras also correspond to non-degenerate involutive set-theoretical solutions of the Yang–Baxter equation, and to certain left brace structures, see [8,1]. A natural generalization of LSA-structures is given by post-Lie algebra structures on pairs of Lie algebras [7].

Étale representations also appear in the classification of adjoint orbits on graded semisimple Lie algebras $\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k$. The classification of G_0-orbits of nilpotent elements can be reduced to determining certain graded semisimple subalgebras $\mathfrak{s}$ associated to such elements which contain an étale representation for the grade-preserving subalgebra $\mathfrak{s}_0$ on the module $\mathfrak{s}_1$, see [19].

It follows from the Whitehead Lemma in Lie algebra cohomology that a semisimple Lie algebra over a field K of characteristic zero does not admit an LSA-structure. The reductive Lie algebra $\mathfrak{gl}_n(K)$, however, admits a canonical LSA-structure. Indeed, it is natural to consider the *reductive* case, where we have the powerful tools of invariant