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Some calculations of the Lusztig–Vogan bijection for classical nilpotent orbits

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ABSTRACT

In this manuscript, we compute explicitly the Lusztig–Vogan bijection for local systems of some classical, special, nilpotent orbits. Using these results, we prove a conjecture of Achar and Sommers on regular functions of some covers of classical nilpotent orbits.

1. Introduction

1.1. Unipotent representations and quantization

Let G be a complex simple Lie group. In [6], Barbasch and Vogan studied the **special unipotent representations** of G, which are of utmost interest in various aspects of representation theory. For instance, they are related to Arthur's packet of automorphic forms, and are conjectured to be unitary. More specifically, they are also conjectured to be 'building blocks' of the unitary dual of G. Indeed, in [4], Barbasch generalized the idea of special unipotent representations to **unipotent representations**, which are used to classify the unitary dual of classical Lie groups.

Another interesting application of special unipotent representations is their relations with the Orbit Method, first introduced by A.A. Kirillov. Roughly speaking, for any

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(co)adjoint orbit $\mathcal{O}$ of a Lie algebra $\mathfrak{g}$, one would like to 'attach' a (preferably unitary) representation to $\mathcal{O}$. This idea is pursued nicely when $\mathfrak{g}$ is a nilpotent or solvable Lie algebra, but several difficulties came up when $\mathfrak{g}$ is semisimple (see [16], [17], [18], [19] for more details). In the context of nilpotent coadjoint orbits in a semisimple Lie algebra $\mathfrak{g}$, the idea of Orbit Method suggests the following:

Conjecture 1.1. *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$, and $R(\mathcal{O})$ be the ring of regular function of $\mathcal{O}$, then there is a (not necessarily unique) $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module Q such that*

$$Q|_{K_{\mathbb{C}}} \cong R(\mathcal{O})$$

(note that $K \leq G$ is the maximal compact subgroup of G, hence its complexification $K_{\mathbb{C}}$ is isomorphic to G). More generally, let $e \in \mathcal{O}$ and G_e be the isotropy group of e with connected component $(G_e)^0$. Then for any irreducible representation ρ of the component group $A(\mathcal{O}) := G_e/(G_e)^0$, there exists a $(\mathfrak{g}_{\mathbb{C}}, K_{\mathbb{C}})$-module Q_ρ such that

$$Q_\rho|_{K_{\mathbb{C}}} \cong R(\mathcal{O}, \rho) = Ind_{G^e}^{G}(\rho),$$

where $R(\mathcal{O}, \rho)$ is the global section of the G-equivariant bundle $G \times_{G_e} V_\rho \to G/G^e \cong \mathcal{O}$. In particular, when $\rho = \mathrm{triv}$ is the trivial representation, then $R(\mathcal{O}, \mathrm{triv}) = R(\mathcal{O})$.

To relate the above conjecture with unipotent representations, recall in [6] that all special unipotent representations of $\mathfrak{g}$ are parametrized by the set

$$\mathcal{N}_{o,\widehat{\overline{a}}} := \{(\mathcal{O}, \pi) \mid \mathcal{O} \text{ is a special nilpotent orbit,} \\ \pi \text{ is an irreducible representation of } \overline{A}(\mathcal{O})\},$$

where $\overline{A}(\mathcal{O})$ is the **Lusztig quotient** of the component group $A(\mathcal{O})$ [6, Section 4]. For each $(\mathcal{O}, \pi) \in \mathcal{N}_{o,\widehat{\overline{a}}}$, we write $X_{\mathcal{O},\pi}$ be its corresponding special unipotent representation. Then we can state a conjecture of Vogan in the context of complex semisimple Lie groups:

Conjecture 1.2 *([18], Conjecture 12.1). Suppose $\mathcal{O}$ is a special nilpotent orbit. For every irreducible representation π of $\overline{A}(\mathcal{O})$, there exists an irreducible representation ρ of $A(\mathcal{O})$ such that*

$$X_{\mathcal{O},\pi}|_{K_{\mathbb{C}}} \cong R(\mathcal{O}, \rho) = Ind_{G^e}^{G}(\rho).$$

For classical nilpotent orbits, Barbasch showed the following:

Theorem 1.3 *([5], Theorem 4.10.1). Let G be a complex simple Lie group of classical type. Then Conjecture 1.2 holds for all special orbits satisfying $A(\mathcal{O}) = \overline{A}(\mathcal{O})$.*