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Characteristic polynomials of symmetric matrices over the univariate polynomial ring

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ABSTRACT

Viewing a bivariate polynomial $f \in \mathbb{R}[x, t]$ as a family of univariate polynomials in t parametrized by real numbers x, we call f *real rooted* if this family consists of monic polynomials with only real roots. If f is the characteristic polynomial of a symmetric matrix with entries in $\mathbb{R}[x]$, it is obviously real rooted. In this article the converse is established, namely that every real rooted bivariate polynomial is the characteristic polynomial of a symmetric matrix over the univariate real polynomial ring. As a byproduct we present a purely algebraic proof of the Helton–Vinnikov Theorem which solved the 60 year old Lax conjecture on the existence of definite determinantal representation of ternary hyperbolic forms.

Introduction

Given a monic polynomial $f \in A[t]$ over a commutative ring A we call a square matrix $M \in \operatorname{Mat}_n A$ a *spectral representation of f over A* if f is the characteristic polynomial of M, i.e., $f = \det(tI_n - M)$. The main result of this paper is the following

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Theorem 1. *Let $f \in \mathbb{R}[x,t]$ be real rooted, i.e., monic in t and for all $a \in \mathbb{R}$ the univariate polynomial $f(a,t) \in \mathbb{R}[t]$ has only real roots. Then f admits a symmetric spectral representation over $\mathbb{R}[x]$, i.e., there exists $M \in \operatorname{Sym}_n \mathbb{R}[x]$ such that $f = \det(tI_n - M)$.*

Symmetric spectral representations as certificates of real rootedness

Given a commutative ring A, it is generally a difficult problem to characterize those monic polynomials $f \in A[t]$ that admit a symmetric spectral representation over A. As noted above, in the case where A is the polynomial ring $\mathbb{R}[x]$ there is an obvious necessary condition, namely that f is real rooted. In other words, this condition means that for every homomorphism $\mathbb{R}[x] \to \mathbb{R}$ the image of f in $\mathbb{R}[t]$ (under coefficient-wise application) has only real roots. The following generalization of this property is shared by all characteristic polynomials of symmetric matrices over any commutative ring A: We call $f \in A[t]$ *real rooted over* A if f is monic and for all ring homomorphisms from A to any real closed field R the image of f in $R[t]$ has only roots in R. In the case $A = \mathbb{R}[x]$ it suffices to check homomorphisms to $\mathbb{R}$ and hence this is indeed a generalization, see Remark 3.2.

Now it is natural to ask about the converse: Which *real rooted* polynomials admit a symmetric spectral representation, or some related, possibly weaker, representation that manifests the real rootedness?

The following characterization of real rooted polynomials over fields is due to Krakowski [16]: If K is any field of characteristic different from 2 then $f \in K[t]$ is real rooted over K if and only if a power of f admits a symmetric spectral representation over K. See also [15] for a generalization and some lower and upper bounds on the exponent needed.

A useful reformulation of the existence of symmetric spectral representations has been given by Bender [3], generalizing a result of Latimer and MacDuffee [22], who established a correspondence between equivalence classes of spectral representations of a polynomial f over the ring of integers $\mathbb{Z}$ and ideal classes in $\mathbb{Z}[t]/(f)$. Bender's observation in [3] serves as an inspiration for the present work as it did for Bass, Estes and Guralnick who proved in [2] that if A is a Dedekind domain and $f \in A[t]$ real rooted, then f divides the characteristic polynomial of a symmetric matrix over A. In other words this means that all roots of f are eigenvalues of a symmetric matrix. Using this result the eigenvalues of adjacency matrices of regular graphs are characterized.

For a slightly smaller class of polynomials, their result can be further extended: A monic polynomial over A is *strictly real rooted* if for any homomorphism $A \to R$ to a real closed field R all roots of the image of f in $R[t]$ lie in R *and are simple*. Kummer recently showed in [19] that for any integral domain A every strictly real rooted polynomial $f \in A[t]$ divides the characteristic polynomial of a symmetric matrix.

The first result towards classification of polynomials that admit symmetric spectral representations without additional factor is also due to Bender [4]: If K is a number field and $f \in K[t]$ is real rooted over K with an odd degree factor, then f admits a symmetric