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Relative tensor triangular Chow groups for coherent algebras

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ABSTRACT

We apply the machinery of relative tensor triangular Chow groups to the action of $\mathbf{D}(\mathrm{Qcoh}(X))$, the derived category of quasi-coherent sheaves on a noetherian scheme X, on the derived category of quasi-coherent $\mathcal{A}$-modules $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$, where $\mathcal{A}$ is a (not necessarily commutative) coherent $\mathcal{O}_X$-algebra. When $\mathcal{A}$ is commutative, we recover the tensor triangular Chow groups of $\mathbf{Spec}(\mathcal{A})$. We also obtain concrete descriptions for integral group algebras and hereditary orders over curves, and we investigate the relation of these invariants to the classical ideal class group of an order. An important tool for these computations is a new description of relative tensor triangular Chow groups as the image of a map in the K-theoretic localization sequence associated to a certain Verdier localization.

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1. Introduction

In [19], Klein defined and began the study of *relative tensor triangular Chow groups*, a family of K-theoretic invariants attached to a compactly generated triangulated category $\mathcal{K}$ with an action of a rigidly-compactly generated tensor triangulated category $\mathcal{T}$ in the sense of [38]. While in [19], they were used to improve upon and extend results of [20], the initial observation of the present work is that they allow us to enter the realm of *noncommutative* algebraic geometry: if X is a noetherian scheme and $\mathcal{A}$ a (possibly noncommutative) coherent $\mathcal{O}_X$-algebra, then the derived category $\mathcal{K} := \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ admits an action by $\mathcal{T} := \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ which is obtained by deriving the tensor product functor

$$\begin{aligned} \mathrm{Qcoh}(\mathcal{A}) \times \mathrm{Qcoh}(\mathcal{O}_X) &\to \mathrm{Qcoh}(\mathcal{A}) \\ (M, F) &\mapsto M \otimes_{\mathcal{O}_X} F. \end{aligned} \tag{1}$$

In this situation, the general machinery of [19] gives us abelian groups $\mathrm{Z}_i^{\Delta}(X, \mathcal{A})$ and $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$, the dimension i tensor triangular cycle and Chow groups of $\mathcal{K}$ relative to the action of $\mathcal{T}$. In the test case where $\mathcal{A}$ is coherent and commutative, and hence $\mathcal{A}$ corresponds to a scheme $\mathbf{Spec}\,\mathcal{A}$ and a finite morphism $\mathbf{Spec}\,\mathcal{A} \to X$, we show that $\mathrm{Z}_i^{\Delta}(X, \mathcal{A})$ and $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$ agree with the dimension i tensor triangular cycle and Chow groups of $\mathbf{D}^{\mathrm{b}}(\mathbf{Spec}\,\mathcal{A})$ as defined in [20], and hence with the usual dimension i cycle and Chow groups of $\mathrm{Z}_i(\mathbf{Spec}\,\mathcal{A})$, $\mathrm{CH}_i(\mathbf{Spec}\,\mathcal{A})$ when $\mathbf{Spec}\,\mathcal{A}$ is a regular algebraic variety (see Theorem 6.6). This computation serves as a motivation to study the groups $\mathrm{Z}_i^{\Delta}(X, \mathcal{A})$ and $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$ for noncommutative coherent $\mathcal{A}$.

We obtain computations of both invariants when $\mathcal{A}$ is a sheaf of hereditary orders on a curve in section 7, and in particular $\mathrm{CH}_i^{\Delta}(X, \mathcal{A})$ recovers the classical *stable class group*