



Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra



Relative tensor triangular Chow groups for coherent algebras



Pieter Belmans^{a,*}, Sebastian Klein

^a *Universiteit Antwerpen, Middelheimlaan 1, Antwerpen, Belgium*

ARTICLE INFO

Article history:

Received 31 January 2017
Available online 7 June 2017
Communicated by Michel Van den Bergh

Keywords:

Chow groups
Tensor triangulated geometry
Maximal orders
Derived categories of sheaves
Noncommutative algebraic geometry

ABSTRACT

We apply the machinery of relative tensor triangular Chow groups to the action of $\mathbf{D}(\mathrm{Qcoh}(X))$, the derived category of quasi-coherent sheaves on a noetherian scheme X , on the derived category of quasi-coherent \mathcal{A} -modules $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$, where \mathcal{A} is a (not necessarily commutative) coherent \mathcal{O}_X -algebra. When \mathcal{A} is commutative, we recover the tensor triangular Chow groups of $\mathbf{Spec}(\mathcal{A})$. We also obtain concrete descriptions for integral group algebras and hereditary orders over curves, and we investigate the relation of these invariants to the classical ideal class group of an order. An important tool for these computations is a new description of relative tensor triangular Chow groups as the image of a map in the K-theoretic localization sequence associated to a certain Verdier localization.

© 2017 Elsevier Inc. All rights reserved.

Contents

1. Introduction	387
2. Tensor triangular preliminaries	388
2.1. Tensor triangular geometry	388
2.2. Supports in large categories	389
2.3. Relative supports and tensor triangular Chow groups	392

* Corresponding author.
E-mail address: pieter.belmans@uantwerpen.be (P. Belmans).

3.	An exact sequence	396
4.	Derived categories of quasi-coherent \mathcal{O}_X -algebras	399
4.1.	Basics of quasi-coherent modules over quasi-coherent \mathcal{O}_X -algebras	399
4.2.	The derived category of a quasi-coherent \mathcal{O}_X -algebra	400
4.2.1.	Basic properties	400
4.2.2.	Taking stalks	401
4.2.3.	Filtrations of the bounded derived category of coherent sheaves	402
5.	Relative tensor triangular Chow groups of a coherent \mathcal{O}_X -algebra	404
5.1.	The action of $\mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ on $\mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$	404
5.2.	Unwinding the definitions	406
5.3.	Comparison to Chow groups of X for coherent \mathcal{O}_X -algebras on regular schemes	410
6.	The case of coherent commutative \mathcal{O}_X -algebras	411
7.	Relative tensor triangular Chow groups for orders	414
7.1.	Preliminaries on orders	414
7.2.	Cycle groups	416
7.3.	Chow groups in the regular case	417
7.4.	Chow groups of (integral) group rings	424
7.5.	Chow groups in the singular case	426
	Acknowledgments	426
	References	427

1. Introduction

In [19], Klein defined and began the study of *relative tensor triangular Chow groups*, a family of K-theoretic invariants attached to a compactly generated triangulated category \mathcal{K} with an action of a rigidly-compactly generated tensor triangulated category \mathcal{T} in the sense of [38]. While in [19], they were used to improve upon and extend results of [20], the initial observation of the present work is that they allow us to enter the realm of *noncommutative* algebraic geometry: if X is a noetherian scheme and \mathcal{A} a (possibly noncommutative) coherent \mathcal{O}_X -algebra, then the derived category $\mathcal{K} := \mathbf{D}(\mathrm{Qcoh}(\mathcal{A}))$ admits an action by $\mathcal{T} := \mathbf{D}(\mathrm{Qcoh}(\mathcal{O}_X))$ which is obtained by deriving the tensor product functor

$$\begin{aligned} \mathrm{Qcoh}(\mathcal{A}) \times \mathrm{Qcoh}(\mathcal{O}_X) &\rightarrow \mathrm{Qcoh}(\mathcal{A}) \\ (M, F) &\mapsto M \otimes_{\mathcal{O}_X} F. \end{aligned} \tag{1}$$

In this situation, the general machinery of [19] gives us abelian groups $Z_i^\Delta(X, \mathcal{A})$ and $\mathrm{CH}_i^\Delta(X, \mathcal{A})$, the dimension i tensor triangular cycle and Chow groups of \mathcal{K} relative to the action of \mathcal{T} . In the test case where \mathcal{A} is coherent and commutative, and hence \mathcal{A} corresponds to a scheme $\mathbf{Spec} \mathcal{A}$ and a finite morphism $\mathbf{Spec} \mathcal{A} \rightarrow X$, we show that $Z_i^\Delta(X, \mathcal{A})$ and $\mathrm{CH}_i^\Delta(X, \mathcal{A})$ agree with the dimension i tensor triangular cycle and Chow groups of $\mathbf{D}^b(\mathbf{Spec} \mathcal{A})$ as defined in [20], and hence with the usual dimension i cycle and Chow groups of $Z_i(\mathbf{Spec} \mathcal{A})$, $\mathrm{CH}_i(\mathbf{Spec} \mathcal{A})$ when $\mathbf{Spec} \mathcal{A}$ is a regular algebraic variety (see Theorem 6.6). This computation serves as a motivation to study the groups $Z_i^\Delta(X, \mathcal{A})$ and $\mathrm{CH}_i^\Delta(X, \mathcal{A})$ for noncommutative coherent \mathcal{A} .

We obtain computations of both invariants when \mathcal{A} is a sheaf of hereditary orders on a curve in section 7, and in particular $\mathrm{CH}_i^\Delta(X, \mathcal{A})$ recovers the classical *stable class group*

Download English Version:

<https://daneshyari.com/en/article/5771973>

Download Persian Version:

<https://daneshyari.com/article/5771973>

[Daneshyari.com](https://daneshyari.com)