Zero loci of Skew-growth functions for dual Artin monoids

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ZERO LOCI OF SKEW-GROWTH FUNCTIONS FOR DUAL ARTIN MONOIDS.

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Abstract. We show that the skew-growth function of a dual Artin monoid of finite type P has exactly $\mathrm{rank}(P) =: l$ simple real zeros on the interval $(0, 1]$. The proofs for types A_l and B_l are based on an unexpected fact that the skew-growth functions, up to a trivial factor, are expressed by Jacobi polynomials due to a Rodrigues type formula in the theory of orthogonal polynomials. The skew-growth functions for type D_l also satisfy Rodrigues type formulae, but the relation with Jacobi polynomials is not straightforward, and the proof is intricate. We show that the smallest root converges to zero as the rank l tends to infinity.

1. Introduction

We study the zero loci of the *skew-growth function* ([Sa2])[1] of a *dual Artin monoid* of finite type ([Be])[2]. The skew-growth function is identified with the generating function of Möbius invariants (called the *characteristic polynomial*) of the *lattice of non-crossing partitions* ([K, Be, B-W]), and is further shown by several authors ([At, A-T, B-W, Ch]) to be equal to the generating function of dimensions of cones of the positive part of the *cluster fan* of Fomin-Zelevinsky ([F-Z1]). We observe that this combinatorially defined function shows an unexpected strong connection with orthogonal polynomials. With the help of them, our goal is to show that the roots of the skew-growth function are simple and lying in the interval $(0, 1]$, and that the smallest root converges to zero as the rank of the type tends to infinity.

Let us explain the contents. Recall ([B-S]) that an Artin group G_M (resp. an Artin monoid G_M^+) associated with a Coxeter matrix $M = (m_{ij})_{i,j\in I}$ is a group (resp. monoid) generated by letters a_i $(i \in I)$ and defined by the Artin braid relations:

$$a_i a_j a_i \cdots \ = \ a_j a_i a_j \cdots \qquad (i, j \in I) \tag{1.1}$$

where both sides are words of alternating sequences of letters a_i and a_j of the same length $m_{ij} = m_{ji} \in \mathbb{Z}_{>0}$ with the initial letters a_i and a_j, respectively. The natural morphism $G_M^+ \to G_M$ is shown to be injective (in particular, G_M^+ is cancellative) so that the Artin monoid is regarded as a submonoid of the Artin group. Requiring more relations $a_i a_i = 1$ $(i \in I)$, we obtain the Coxeter group $\overline{G}_M$ and the quotient morphism: $\pi : G_M \to \overline{G}_M$ (for short, we shall denote $\pi(g)$ by $\overline{g}$ for $g \in G_M$). We call $\mathbf{S} := \{a_i \mid i \in I\}$ the simple generator system of the Artin group and

[1] The study of the skew-growth function of a monoid and its zero loci is motivated by the study of the *partition functions associated with the monoid*, since the partition functions are given by certain residue formula at the zero loci of the skew-growth function ([Sa3] §11 Th.6 and §12).

[2] Dual Artin monoids were introduced by D. Bessis ([Be], c.f. [B-K-L]) under the name *dual braid monoids*. For more detailed explanations, see the following part of §1 and Appendix III.