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Quantifying residual finiteness of linear groups

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ABSTRACT

Normal residual finiteness growth measures how well a finitely generated residually finite group is approximated by its finite quotients. We show that any finitely generated linear group $\Gamma \leq \mathrm{GL}_d(K)$ has normal residual finiteness growth asymptotically bounded above by $(n \log n)^{d^2-1}$; notably this bound depends only on the degree of linearity of Γ. If char $K = 0$ or K is a purely transcendental extension of a finite field, then this bound can be improved to n^{d^2-1}. We also give lower bounds on the normal residual finiteness growth of Γ in the case that Γ is a finite index subgroup of $G(\mathbb{Z})$ or $G(\mathbb{F}_p[t])$, where G is Chevalley group of rank at least 2. These lower bounds agree with the computed upper bounds, providing exact asymptotics on the normal residual finiteness growth. In particular, finite index subgroups of $G(\mathbb{Z})$ and $G(\mathbb{F}_p[t])$ have normal residual finiteness growth $n^{\dim(G)}$. We also compute the non-normal residual finiteness growth in the above cases; for the lower bounds the exponent $\dim(G)$ is replaced by the minimal codimension of a maximal parabolic subgroup of G.

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1. Introduction

Let Γ be a finitely generated residually finite group with finite generating set X. If $\gamma \in \Gamma$, let $||\gamma||_X$ be the word length of γ with respect to X. If $\gamma \in \Gamma$ is nontrivial, we say a finite quotient Q of Γ detects γ if the image of γ in Q is nontrivial. Define $F^{\trianglelefteq}_{\Gamma,X}(n)$ to be the smallest natural number N such that for all $\gamma \in \Gamma$ with $||\gamma||_X \leq n$, γ is detected by a quotient of size at most N.

We call the function $F^{\trianglelefteq}_{\Gamma,X}(n)$ the normal residual finiteness growth function of Γ. This function was first studied by Bou-Rabee in [1], and its asymptotics have been studied for virtually nilpotent linear groups [2], arithmetic groups [3], linear groups [4], and free groups [5,6], with the best current estimate for free groups given in [7]. A related function is $F^{\leq}_{\Gamma,X}(n)$, the non-normal residual finiteness growth function of Γ, defined as the smallest natural number N such that for all $\gamma \in \Gamma$ with $||\gamma||_X \leq n$, there exists $H \leq G$ with $\gamma \notin H$ and $[G : H] \leq N$. This function has also been studied for certain classes of groups, in particular for virtually special groups in [8] and for free groups in [2,9,10]. Our goal in this paper is to obtain better estimates of the functions $F^{\trianglelefteq}_{\Gamma,X}(n)$ and $F^{\leq}_{\Gamma,X}(n)$ when Γ is a linear group.

While these functions depend on the choice of generating set X, their asymptotic growths, which we call the normal residual finiteness growth of Γ and non-normal residual finiteness growth of Γ, respectively, are independent of the choice of generating set ([1], Lemma 1.1). We thus drop the reference to X for the remainder of the introduction. We compare the asymptotic growth of functions by writing $f \preceq g$ if for some C, $f(n) \leq Cg(Cn)$ for all n.

It was shown in [4] that if Γ is a finitely generated linear group over an infinite field, then $F^{\trianglelefteq}_{\Gamma}(n) \preceq n^k$ for some k depending on the field and the degree of linearity. A natural question is whether the dependence on the field of coefficients is necessary. Our first result is that in fact there is a uniform bound on the residual finiteness growth of finitely generated linear groups with a fixed degree of linearity.

Theorem 1.1. *Let $\Gamma \leq \mathrm{GL}_d(K)$ be a finitely generated linear group with $d \geq 2$.*

(i) $F^{\trianglelefteq}_{\Gamma}(n) \preceq (n \log n)^{d^2-1}$ *and* $F^{\leq}_{\Gamma}(n) \preceq (n \log n)^{d-1}$.
(ii) *If char $K = 0$ or K is a purely transcendental extension of a finite field, then* $F^{\trianglelefteq}_{\Gamma}(n) \preceq n^{d^2-1}$ *and* $F^{\leq}_{\Gamma}(n) \preceq n^{d-1}$.

One potential application of normal residual finiteness growth is in showing a group is nonlinear. For a finitely generated group Γ, one can show $F_{\Gamma}(n)$ is super-polynomial to conclude Γ is nonlinear. If Γ is infinitely generated, the uniform bound of Theorem 1.1 provides another method for establishing nonlinearity. In particular, this result has potential applications in the study of profinite groups.