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Mesoprimary decomposition of binomial submodules

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ABSTRACT

Recent results of Kahle and Miller give a method of constructing primary decompositions of binomial ideals by first constructing "mesoprimary decompositions" determined by their underlying monoid congruences. Mesoprimary decompositions are highly combinatorial in nature, and are designed to parallel standard primary decomposition over Noetherean rings. In this paper, we generalize mesoprimary decomposition from binomial ideals to "binomial submodules" of certain graded modules over a monoid algebra, analogous to the way primary decomposition of ideals over a Noetherean ring R generalizes to R-modules. The result is a combinatorial method of constructing primary decompositions that, when restricting to the special case of binomial ideals, coincides with the method introduced by Kahle and Miller.

1. Introduction

Fix a field $\Bbbk$ and a commutative monoid Q. A *binomial ideal* in the monoid algebra $\Bbbk[Q]$ is an ideal I whose generators have at most two terms. The quotient $\Bbbk[Q]/I$ by a

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binomial ideal identifies, up to scalar multiple, any monomials appearing in the same binomial in I. This induces a *congruence* $\sim_I$ on the monoid Q (an equivalence relation perserving additivity), and the quotient module $\Bbbk[Q]/I$ is naturally graded with a decomposition into 1-dimensional $\Bbbk$-vector spaces, at most one per $\sim_I$-class. In [4], Kahle and Miller introduce *mesoprimary decompositions*, which are combinatorial approximations of primary decompositions of I constructed from the congruence $\sim_I$.

Mesoprimary decomposition of binomial ideals is motivated by combinatorially constructed primary decompositions of monomial ideals. Any monomial ideal I in the monoid algebra $\Bbbk[Q]$ is uniquely determined by the monomials it contains. Taking the quotient $\Bbbk[Q]/I$ amounts to setting these monomials to 0, and the monomials that lie outside of I naturally grade the quotient $\Bbbk[Q]/I$ with a decomposition into 1-dimensional $\Bbbk$-vector spaces.

An irreducible decomposition for a monomial ideal I whose components are themselves monomial ideals can be constructed by locating *witness monomials* $\mathbf{x}^w$ whose annihilator modulo I is prime, and then constructing for each witness monomial $\mathbf{x}^w$ the primary monomial ideal that contains all monomials not lying below $\mathbf{x}^w$. The intersection of these ideals (one per witness monomial) equals I, and the witnesses are readily identified from the grading on $\Bbbk[Q]/I$. See [8, Chapter 5] for a full treatment of monomial irreducible decomposition.

Combinatorially constructed irreducible decompositions of monomial ideals have also been shown to live within a larger categorical setting. Much in the way primary decomposition of ideals over a Noetherean ring R generalizes to R-modules, combinatorial methods for constructing primary decompositions of monomial ideals can be generalized to certain modules whose gradings resemble the fine gradings of monomial quotients. See [7] for an overview of these constructions and [3,6] for consequences.

Kahle and Miller use congruences to extend the above construction from monomial ideals to binomial ideals [4]. Given a binomial ideal I, they pinpoint a collection of monomials in $\Bbbk[Q]/I$ that behave like witnesses. For each witness $\mathbf{x}^w$, they construct the *coprincipal component* at $\mathbf{x}^w$, a binomial ideal containing I whose quotient has $\mathbf{x}^w$ as the unique greatest nonzero monomial. The resulting collection of ideals, one for each witness, decomposes I, and each component admits a canonical primary decomposition. In this way, mesoprimary decompositions act as a bridge to primary components of a binomial ideal from the combinatorics of its induced congruence.

Mesoprimary decompositions are constructed in two settings: first for monoid congruences, and then for binomial ideals; both are designed to parallel standard primary decomposition in a Noetherian ring R. This motivated Kahle and Miller to pose Problems 1.1 and 1.2 below, which appeared as [4, Problem 17.11] and [4, Problem 17.13], respectively. These problems, in turn, serve to motivate the results in this paper.