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Representing finitely generated refinement monoids as graph monoids ☆

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Graph monoids arise naturally in the study of non-stable K-theory of graph C*-algebras and Leavitt path algebras. They play also an important role in the current approaches to the realization problem for von Neumann regular rings. In this paper, we characterize when a finitely generated conical refinement monoid can be represented as a graph monoid. The characterization is expressed in terms of the behavior of the structural maps of the associated I-system at the free primes of the monoid.

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1. Introduction

The class of commutative monoids satisfying the Riesz refinement property – refinement monoids for short – has been largely studied over the last decades in connection with various problems such as non-stable K-Theory of rings and C^*-algebras (see e.g. [4,10,11,19,24]), classification of Boolean algebras (see e.g. [20,25]), or its own structure theory (see e.g. [16,17,28]).

An important invariant in non-stable K-theory is the commutative monoid $\mathcal{V}(R)$ associated to any ring R, consisting of the isomorphism classes of finitely generated projective (left, say) R-modules, with the operation induced from direct sum. If R is a (von Neumann) regular ring or a C*-algebra with real rank zero (more generally, an exchange ring), then $\mathcal{V}(R)$ is a refinement monoid (e.g., [10, Corollary 1.3, Theorem 7.3]).

The *realization problem* asks which refinement monoids appear as a $\mathcal{V}(R)$ for R in one of the above-mentioned classes. Wehrung [29] constructed a conical refinement monoid of cardinality $\aleph_2$ which is not isomorphic to $\mathcal{V}(R)$ for any regular ring R, but it is an important open problem, appearing for the first time in [18], to determine whether every countable conical refinement monoid can be realized as $\mathcal{V}(R)$ for some regular ring R. See [3] for a survey on this problem, and [9] for some recent progress on the problem, with connections with the Atiyah Problem.

An interesting situation in which the answer to the realization problem is affirmative is the following:

Theorem 1.1 *([5, Theorem 4.2, Theorem 4.4]). Let E be a row-finite graph, let $M(E)$ be its graph monoid, and let K be any field. Then there exists a (not necessarily unital) von Neumann regular K-algebra $Q_K(E)$ such that $\mathcal{V}(Q_K(E)) \cong M(E)$.*

Thus, an intermediate step that could be helpful to give an answer to the realization problem is to characterize which conical refinement monoids are representable as graph monoids. The first author, Perera and Wehrung gave such a characterization in the concrete case of finitely generated antisymmetric refinement monoids [13, Theorem 5.1]. These monoids are a particular case of primely generated refinement monoids (see e.g. [16]). Recall that an element p in a monoid M is a *prime element* if p is not invertible in M, and, whenever $p \leq a + b$ for $a, b \in M$, then either $p \leq a$ or $p \leq b$ (where $x \leq y$ means that $y = x + z$ for some $z \in M$). The monoid M is *primely generated* if every non-invertible element of M can be written as a sum of prime elements. Primely generated refinement monoids enjoy important cancellation properties, such as separative cancellation and unperforation, as shown by Brookfield in [16, Theorem 4.5, Corollary 5.11(5)]. Moreover, it was shown by Brookfield that any finitely generated refinement monoid is automatically primely generated [16, Corollary 6.8].

In [12], the authors of the present paper showed that any primely generated refinement monoid can be represented, up to isomorphism, as the monoid associated to an I-system