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Ascent and descent of the Golod property along algebra retracts

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ABSTRACT

We study ascent and descent of the Golod property along an algebra retract. We characterise trivial extensions of modules, fibre products of rings to be Golod rings. We present a criterion for a graded module over a graded affine algebra of characteristic zero to be a Golod module.

1. Introduction

Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$. Let M be a finitely generated R-module. The generating function of the sequence of Betti numbers of the minimal free resolution of M over R is a formal power series in $\mathbb{Z}[|t|]$. This series is called the Poincaré series of M over R and is denoted by $P_M^R(t)$ (Definition 2.1). J-P. Serre showed that this series is coefficient-wise bounded above by a series represent-

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ing a rational function. The module M is said to be a Golod module when the Poincaré series coincides with the upper bound given by Serre (Definition 2.4). The ring R is said to be a Golod ring if its residue field k is a Golod R-module. We refer the reader for details regarding Golod rings and Golod modules to the survey article [6] by Avramov. The main objectives of this article are to study transfer of the Golod property along algebra retracts and more generally large homomorphisms, to establish a connection between the Golod property of a module and its trivial extension and finally to characterise the Golod property of fibre products of local rings.

A subring of a ring is called an algebra retract if the inclusion map has a left inverse (Definition 2.8). Several authors have studied how ring-theoretic properties transfer along algebra retracts from different perspectives. Basic properties like normality of domains, semi-normality, regularity, complete intersection, Koszul, Stanley–Reisner are known to descend along algebra retracts (see [2,9,8,24]). On the other hand, properties like Cohen–Macaulay, Gorenstein are not inherited by an algebra retract in general. We refer the reader to [9] for a very good exposition on this theme. In the present article we prove the following:

Theorem 1.1. *Let $j : (R, \mathfrak{m}) \to (A, \mathfrak{n})$ be an algebra retract with a section $p : (A, \mathfrak{n}) \to (R, \mathfrak{m})$. Let M be a finitely generated R-module which is Golod when viewed as an A-module via the homomorphism p. Then M is also a Golod R-module.*

The Golod property does not ascend along an algebra retract in general as seen by any non-Golod local ring containing its residue field. So certain assumptions are necessary for an affirmative answer. Our main result stated below presents one such assumption.

Theorem 1.2. *Let $j : (R, \mathfrak{m}) \to (A, \mathfrak{n})$ be an algebra retract which admits sections (possibly equal) p and p'. Let $\ker(p) = I$ and $\ker(p') = I'$ satisfy $II' = 0$. Consider R-module structures on I, I' via the retract map j. Then the ideal I is a Golod R-module if and only if I' is so.*

Let N be an R-module. If we consider N as an A-module via any of the maps p, p', then N is a Golod A-module if and only if N is a Golod R-module and I (equivalently I') is a Golod R-module. In particular, A is a Golod ring if and only if I (equivalently I') is a Golod R-module.

As an application we prove the following theorem.

Theorem 1.3. *Let $(R, \mathfrak{m})$ be a local ring and M an R-module. Let $A = R \ltimes M$ be the trivial extension of R by M. Then A is a Golod ring if and only if M is a Golod R-module.*

The above result gives us an efficient method to study the Golod property of modules with results available to characterise Golod rings. We demonstrate this by giving a new characterisation of regularity of a ring in terms of the Golod property of its canonical module (Corollary 5.3).