Asymptotic Weights of Syzygies of Toric Varieties

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0. Abstract

The purpose of the paper is to give a sharp asymptotic description of the weights that appear in the syzygies of a smooth toric variety. We prove that as the positivity of the embedding increases, in any strand of syzygies, torus weights after normalization stabilize to the same fixed shape that we explicitly specify.

1. Introduction

The purpose of the paper is to give a sharp asymptotic description of the weights that appear in the syzygies of a smooth toric variety. We prove that as the positivity of the embedding increases, in any strand of syzygies, torus weights after normalization stabilize to the same fixed shape that we explicitly specify.

Let X be a smooth projective toric variety over $\mathbb{C}$ of dimension n throughout the paper, and L be a very ample toric line bundle on X. Then L defines a toric embedding:

$$X \hookrightarrow \mathbb{P}^{r(L)} = \mathbb{P}H^0(X, L) = \text{Proj } S$$

where $r(L) = h^0(X, L) - 1$ and $S = \text{Sym}H^0(X, L)$. Write:

$$R(X; L) = \bigoplus_m H^0(X, mL)$$

which is viewed as a finitely generated graded S-module. We will be interested in the syzygies of $R(X; L)$ over S. Specifically, R has a graded minimal free resolution

$$\mathbb{F} : ... \to F_p \to ... \to F_0 \to R \to 0$$

where $F_p = \oplus_j S(-a_{p,j})$ is a free S-module. Write $K_{p,q}(X; L)$ for the finite dimensional vector space of minimal p-th syzygies of degree $(p + q)$, so that:

$$F_p \cong \bigoplus_q K_{p,q}(X; L) \otimes_{\mathbb{C}} S(-p - q)$$

Moreover, in the above setting, the torus action on X induces torus actions on $K_{p,q}(X; L)$. We can naturally ask which torus weights appear in their decompositions.

From an asymptotic perspective, Ein and Lazarsfeld show in [EL11] that for $1 \leq q \leq n$, if $L \gg 0$, $K_{p,q}(X; L) \neq 0$ for almost all $p \in [1, r_d]$. In this paper, we give a sharp description of the asymptotic distribution of normalized torus weights in syzygies. To give the statement, let Δ be the convex polytope associated to the very ample divisor A ([F93], Section 3.4, p66, P_A in notation of the book.) Let $L_d = A^{\otimes d}$. Then by degree counting, the torus weights of $K_{p,q}(X; L_d)$ correspond to integral points in $(p + q)d \cdot \Delta$. Denote the collection of weights by:

$$\text{wts}(K_{p,q}(X; L_d)) = \{\text{Torus weights of } K_{p,q}(X; L_d)\} \subseteq (p + q)d \cdot \Delta$$