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Coherence, subgroup separability, and metacyclic structures for a class of cyclically presented groups

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A B S T R A C T

We study a class $\mathfrak{M}$ of cyclically presented groups that includes both finite and infinite groups and is defined by a certain combinatorial condition on the defining relations. This class includes many finite metacyclic generalized Fibonacci groups that have been previously identified in the literature. By analyzing their shift extensions we show that the groups in the class $\mathfrak{M}$ are coherent, subgroup separable, satisfy the Tits alternative, possess finite index subgroups of geometric dimension at most two, and that their finite subgroups are all metacyclic. Many of the groups in $\mathfrak{M}$ are virtually free, some are free products of metacyclic groups and free groups, and some have geometric dimension two. We classify the finite groups that occur in $\mathfrak{M}$, giving extensive details about the metacyclic structures that occur, and we use this to prove an earlier conjecture concerning cyclically presented groups in which the relators are positive words of length three. We show that any finite group in the class $\mathfrak{M}$ that has fixed point free shift automorphism must be cyclic.

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1. Introduction

Given a positive integer n, let F be the free group with basis $x_0, \dots, x_{n-1}$ and let $\theta : F \to F$ be the *shift automorphism* given by $\theta(x_i) = x_{i+1}$ with subscripts modulo n. Given a word w representing an element of F, the group

$$G_n(w) = \langle x_0, \dots, x_{n-1} \mid w, \theta(w), \dots, \theta^{n-1}(w) \rangle$$

is called a *cyclically presented group*. The shift defines an automorphism of $G_n(w)$ with exponent n and the resulting $\mathbb{Z}_n$-action on $G_n(w)$ determines the *shift extension* $E_n(w) = G_n(w) \rtimes_\theta \mathbb{Z}_n$, which admits a two-generator two-relator presentation of the form

$$E_n(w) = \langle t, x \mid t^n, W \rangle \tag{1}$$

where $W = W(x,t)$ is obtained by rewriting w in terms of the substitutions $x_i = t^i x t^{-i}$ (see, for example, [32, Theorem 4]). The shift on $G_n(w)$ is then realized by conjugation by t in $E_n(w)$. The group $G_n(w)$ is recovered as the kernel of a retraction $\nu^0 : E_n(w) \to \mathbb{Z}_n = \langle t \mid t^n \rangle$ that trivializes x.

In the fifty years since Conway's question in [20], the systematic study of cyclically presented groups has led to the introduction and study of numerous families of cyclic presentations with combinatorial structure determined by various parameters. These include the *Fibonacci groups* $F(2,n) = G_n(x_0 x_1 x_2^{-1})$ [20] and a host of generalizing families including the groups $F(r,n)$ [32], $F(r,n,k)$ [11], $R(r,n,k,h)$ [14], $H(r,n,s)$ [15], $F(r,n,k,s)$ [16] (see also, for example, [48]), $P(r,n,l,s,f)$ [42,53], $H(n,m)$ [25], $G_n(m,k) = G_n(x_0 x_m x_k^{-1})$ [31,17,1,52], and the groups $G_n(x_0 x_k x_l)$ [18, 23,7].

A number of these articles identify infinite families of cyclic presentations that define finite metacyclic groups. One purpose of this article is to give a unified explanation for results of this type that appeared in [13–15,23,32], see Corollaries 9–12 below. Our approach leads to the identification of new infinite families of cyclic presentations that define finite metacyclic groups, including but not limited to the class of presentations treated in [42], see Corollary 8 and Corollary B.

All of the cyclic presentations discussed in the preceding paragraph have a common combinatorial form. Given integers f and $r \geq 0$, let

$$\Lambda(r,f) = \prod_{i=0}^{r-1} x_{if} \tag{2}$$

be the positive word of length r consisting of equally spaced generators, starting with x_0 and having *step size* f (where an empty product denotes the empty word). We call any shift of $\Lambda(r,f)$ or its inverse an f-*block* and we say that a cyclically presented group G is of *type* $\mathfrak{F}$ if there is an integer f such that $G \cong G_n(w)$ where w is a product of two