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The pro-nilpotent group topology on a free group

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ABSTRACT

In this paper, we study the pro-nilpotent group topology on a free group. First we describe the closure of the product of finitely many finitely generated subgroups of a free group in the pro-nilpotent group topology and then present an algorithm to compute it. We deduce that the nil-closure of a rational subset of a free group is an effectively constructible rational subset and hence has decidable membership. We also prove that the $\mathsf{G}_{\mathsf{nil}}$-kernel of a finite monoid is computable and hence pseudovarieties of the form $\mathsf{V} \,ⓜ\, \mathsf{G}_{\mathsf{nil}}$ have decidable membership problem, for every decidable pseudovariety of monoids V. Finally, we prove that the semidirect product $\mathsf{J} * \mathsf{G}_{\mathsf{nil}}$ has a decidable membership problem.

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1. Introduction

Hall showed that every finitely generated subgroup of a free group is closed in the profinite topology [10]. Pin and Reutenauer conjectured that if $H_1, H_2, \ldots, H_n$ are finitely generated subgroups of a free group, then the product $H_1 H_2 \cdots H_n$ is closed in the profinite topology [16]. Assuming this conjecture they presented a simple algorithm to compute the closure of a given rational subset of a free group. This conjecture was proved by Ribes and Zalesskiĭ [18], who also later proved that if the subgroups $H_1, \ldots, H_n$ are p-closed for some prime p, then $H_1 \cdots H_n$ is p-closed, too [19].

Margolis, Sapir and Weil provided an algorithm to compute the nil-closure of a finitely generated subgroup of a free group [15]. In this paper, we provide an example of two finitely generated nil-closed subgroups H, K of a free group whose product HK is not nil-closed. However, we prove that the nil-closure of the product of finitely many finitely generated subgroups is the intersection over all primes p of its p-closures and present a procedure to compute algorithmically a finite automaton that accepts precisely the reduced words in the nil-closure of the product. Hence, there is a uniform algorithm to compute membership in the nil-closure of a product of finitely many finitely generated subgroups of a free group. We also prove that the nil-closure of a rational subset of a free group is again a rational subset and provide an algorithm to compute an automaton that accepts the reduced words in the nil-closure. This yields that the Mal'cev product $\mathsf{V} ⓜ \mathsf{G}_{\mathsf{nil}}$ is decidable for every decidable pseudovariety of monoids V, where $\mathsf{G}_{\mathsf{nil}}$ is the pseudovariety of all finite nilpotent groups.

Auinger and the third author introduced the concept of arboreous pseudovarieties of groups [4] and proved that a pseudovariety of groups H is arboreous if and only if $\mathsf{J} ⓜ \mathsf{H} = \mathsf{J} * \mathsf{H}$, where J is the pseudovariety of all finite $\mathcal{J}$-trivial monoids. The pseudovariety $\mathsf{G}_{\mathsf{nil}}$ is not arboreous and, therefore, $\mathsf{J} ⓜ \mathsf{G}_{\mathsf{nil}} \neq \mathsf{J} * \mathsf{G}_{\mathsf{nil}}$. We prove that the pseudovariety $\mathsf{J} * \mathsf{G}_{\mathsf{nil}}$ has decidable membership as an application of our results on computing nil-closures of rational subsets.

2. Decidability of the nil-closure of rational subsets

Let G be a group and H a pseudovariety of groups, that is, a class of finite groups closed under finite direct products, subgroups and homomorphic images. Then the pro-H topology on G is the group topology defined by taking as a fundamental system of neighborhoods of the identity all normal subgroups N of G such that $G/N \in \mathsf{H}$. This is the weakest topology on G so that every homomorphism of G to a group in H (endowed with the discrete topology) is continuous. We say that G is *residually* H if, for every $g \in G \setminus \{1\}$, there is a homomorphism $\varphi\colon G \to H \in \mathsf{H}$ with $\varphi(g) \neq 1$, or, equivalently, $\{1\}$ is an H-closed subgroup. In this case, the pro-H topology is Hausdorff and, in fact, it is metric when G is finitely generated. More precisely, when G is finitely generated, the topology is given by the following ultrametric écart. For $g \in G$, define